Math 631: Problem Set 4

Due Friday October 3 , 2008

1. Morphisms of quasi-projective varieties. Recall that by defintion, a morphism of quasiprojective varieties is a map $\phi : X \to Y$ (where say $X \subset \mathbf{P}^n$ and $Y \in \mathbf{P}^m$) such that every $x \in X$ admits a non-empty open neighborhood U such that $\phi(U) \subset \mathcal{U}_i$ for some standard affine chart $\mathcal{U}_i$ of $\mathbf{P}^m$ and

$$\phi|_U : U \to Y \cap \mathcal{U}_i \subset \mathcal{U}_i = \mathbb{A}^m$$

is given by

$$y \mapsto (\phi_1(y), \ldots, \phi_m(y))$$

where

$$\phi_i \in \mathcal{O}_X(U).$$

Prove that this definition is equivalent to each of the following:

a). For each point x in X, there is a neighborhood of x where ϕ agrees with (the restriction of) a polynomial mapping $\mathbf{P}^n \dashrightarrow \mathbf{P}^m$ given by $m+1$ homogeneous polynomials of the same degree in $n+1$ variables, $[x_0 : \ldots : x_n] \mapsto [F_0(x) : \ldots : F_m(x)]$.

b). For any affine covers $\{U_\lambda\}$ of X and $\{V_\mu\}$ of Y such that each $\phi(U_\lambda)$ is contained in some V_μ, we have that $\phi|_{U_\lambda}$ is a regular map (in the sense of affine varieties) from U_λ to V_μ.

Note that Definition b makes sense even for abstract varieties (as defined on problem set 2), whereas for many problems, Definition a turns out to be the easiest to work with.

2. Compliments of Hypersurfaces are affine. A hypersurface in $\mathbf{P}^n$ is a Zariski closed set defined by a single (homogeneous) polynomial. If H is a hypersurface in $\mathbf{P}^n$, show that $\mathbf{P}^n - H$ is affine. (Hint: Do the case where H is hyperplane first, and think about the Veronese map to get the general case.)

3. Hyperplane through general points. Fix any n points in $\mathbf{P}^n$. Prove that there is a hyperplane containing these n points, and that if the n points are in "general position," then there is a unique hyperplane containing them all. Explain the meaning of "general position" in the context of this problem. For example, when n is two, it is clear that two points determine a line; general position here means the points are distinct.[1]

4. Hypersurfaces through points. a). Fix a point $P \in \mathbf{P}(V) = \mathbf{P}^n$. Show that the set of hypersurfaces of degree d in $\mathbf{P}(V)$ passing through P is naturally parametrized by a hyperplane in the variety $\mathbf{P}(Sym^d(V^*))$ parametrizing all degree d hypersurfaces in $\mathbf{P}(V)$.

b). Fix a natural number d. Find q such that following sentence is meaningful (and true): "Through q general points in the projective plane, there passes a uniquely determined curve (ie, hypersurface in $\mathbf{P}^2$) of degree d."

[1] Caution: the meaning of the ubiqituous phrase "general position" in algebraic geometry varies depending on the context, even on this very problem set!

5. A non-affine, non-projective variety. Let X be the quasi-projective variety $\mathbb{A}^2 - \{(0,0)\}$. Find a simple presentation of the ring of regular functions on X (and prove it!). Use this to show that X can not be affine.

6. Every projective variety is defined by Quadrics. Let X be an arbitrary projective variety in $\mathbf{P}^n$.

a). Show that X can be described as the common zero set of a collection of homogeneous polynomials having all the same degree.

b). Show that X is isomorphic to a "linear section" of a Veronese n-fold. That is, there is some d such that X is isomorphic to a variety of the form $V_d \cap L$ where L is a linear variety in $\mathbf{P}^N$ and V_d is the image of $\mathbf{P}^n$ under the Veronese map ν_d.

c). Show that X is isomorphic to an intersection of quadrics. (A quadric is a hypersurface in projective space defined by a homogeneous polynomial of degree two.)

7. Grassmannians. Fix a vector space V of finite dimension n, over an arbitrary ground field k. Let $\mathbb{G}_d(V)$ denote the set of all d-dimensional subspaces of V. Thus $\mathbb{G}_1(V) = \mathbf{P}(V)$.

a). Explain how the set of lines in $\mathbf{P}(V)$ is naturally identified with the set $\mathbb{G}_2(V)$, and that in general, the linear spaces in $\mathbf{P}(V)$ of dimension d are the points in $\mathbb{G}_{d+1}(V)$.

b). Fix a basis for V, thereby identifying V with k^n. Use this to represent any d-dimensional subspace W as a $d \times n$ matrix of scalars, of full rank d.

c). Prove that two rank d matrices A_1 and A_2 of size $d \times n$ determine the same subspace W if and only if there exists an element $g \in GL(d,k)$ such that $A_1 = gA_2$. Conclude that the set $\mathbb{G}_d(V)$ can be identified (as a set) with the the quotient of the set of all full rank $d \times n$ matrices by the natural action of $GL(d)$ on the left.[2]

d). Show that $\mathbb{G}_d(V)$ is covered by $\binom{n}{d}$ sets U_I, each identified with affine space $\mathbb{A}^{d(n-d)}$ in a natural way. In the case $d = 1$, your cover should specialize to the standard affine cover of $\mathbf{P}(V)$.

e). Let U_1 and U_2 be two of the sets identified in (d), and let V_i be the subset of $\mathbb{A}^{d(n-d)}$ corresponding to $U_1 \cap U_2$ under the identifications $\phi_i : U_i \longrightarrow \mathbb{A}^{d(n-d)}$. Prove that V_i is open in $\mathbb{A}^{d(n-d)}$, and explicitly describe the chart change map $\phi_2 \circ \phi_1^{-1} : V_1 \longrightarrow V_2$.

f). Explain how $\mathbb{G}_d(V)$ is an abstract variety[3] over any field k, a smooth manifold over $\mathbb{R}$, and a complex manifold over $\mathbb{C}$. What is its dimension?

[2]If the ground field is, say $\mathbb{R}$, this endows $\mathbb{G}_d(V)$ with a natural structure as a topological space, which agrees with the Hausdorff topology on $\mathbf{P}(V)$ in the case $d = 1$.

[3]Even better, on the next problem set, we will see that $\mathbb{G}_d(V)$ has an embedding into projective space which gives it the structure of a *projective variety*.