Math 631: Problem Set 4

Due Friday October 3, 2008

1. **Morphisms of quasi-projective varieties.** Recall that by definition, a morphism of quasiprojective varieties is a map \( \phi : X \to Y \) (where say \( X \subset \mathbb{P}^n \) and \( Y \subset \mathbb{P}^m \)) such that every \( x \in X \) admits a non-empty open neighborhood \( U \) such that \( \phi(U) \subset U_i \) for some standard affine chart \( U_i \) of \( \mathbb{P}^m \) and

\[
\phi|_U : U \to Y \cap U_i \subset U_i = \mathbb{A}^m
\]

is given by

\[
y \mapsto (\phi_1(y), \ldots, \phi_m(y))
\]

where

\[
\phi_i \in \mathcal{O}_X(U).
\]

Prove that this definition is equivalent to each of the following:

a). For each point \( x \) in \( X \), there is a neighborhood of \( x \) where \( \phi \) agrees with (the restriction of) a polynomial mapping \( \mathbb{P}^n \to \mathbb{P}^m \) given by \( m+1 \) homogeneous polynomials of the same degree in \( n+1 \) variables, \( x_0 : \ldots : x_n \mapsto [F_0(x) : \ldots : F_m(x)] \).

b). For any affine covers \( \{U_\lambda\} \) of \( X \) and \( \{V_\mu\} \) of \( Y \) such that each \( \phi(U_\lambda) \) is contained in some \( V_\mu \), we have that \( \phi|_{U_\lambda} \) is a regular map (in the sense of affine varieties) from \( U_\lambda \) to \( V_\mu \).

Note that Definition b makes sense even for abstract varieties (as defined on problem set 2), whereas for many problems, Definition a turns out to be the easiest to work with.

2. **Compliments of Hypersurfaces are affine.** A hypersurface in \( \mathbb{P}^n \) is a Zariski closed set defined by a single (homogeneous) polynomial. If \( H \) is a hypersurface in \( \mathbb{P}^n \), show that \( \mathbb{P}^n - H \) is affine. (Hint: Do the case where \( H \) is hyperplane first, and think about the Veronese map to get the general case.)

3. **Hyperplane through general points.** Fix any \( n \) points in \( \mathbb{P}^n \). Prove that there is a hyperplane containing these \( n \) points, and that if the \( n \) points are in “general position,” then there is a unique hyperplane containing them all. Explain the meaning of “general position” in the context of this problem. For example, when \( n \) is two, it is clear that two points determine a line; general position here means the points are distinct.\(^1\)

4. **Hypersurfaces through points.** a). Fix a point \( P \in \mathbb{P}(V) = \mathbb{P}^n \). Show that the set of hypersurfaces of degree \( d \) in \( \mathbb{P}(V) \) passing through \( P \) is naturally parametrized by a hyperplane in the variety \( \mathbb{P}(\text{Sym}^d(V^*)) \) parametrizing all degree \( d \) hypersurfaces in \( \mathbb{P}(V) \).

b). Fix a natural number \( d \). Find \( q \) such that following sentence is meaningful (and true): “Through \( q \) general points in the projective plane, there passes a uniquely determined curve (ie, hypersurface in \( \mathbb{P}^2 \)) of degree \( d \).”

\(^1\)Caution: the meaning of the ubiquitous phrase “general position” in algebraic geometry varies depending on the context, even on this very problem set!
5. A non-affine, non-projective variety. Let $X$ be the quasi-projective variety $\mathbb{A}^2 - \{(0,0)\}$. Find a simple presentation of the ring of regular functions on $X$ (and prove it!). Use this to show that $X$ can not be affine.

6. Every projective variety is defined by Quadrics. Let $X$ be an arbitrary projective variety in $\mathbb{P}^n$.

a). Show that $X$ can be described as the common zero set of a collection of homogeneous polynomials having all the same degree.

b). Show that $X$ is isomorphic to a “linear section” of a Veronese $n$-fold. That is, there is some $d$ such that $X$ is isomorphic to a variety of the form $V_d \cap L$ where $L$ is a linear variety in $\mathbb{P}^N$ and $V_d$ is the image of $\mathbb{P}^n$ under the Veronese map $\nu_d$.

c). Show that $X$ is isomorphic to an intersection of quadrics. (A quadric is a hypersurface in projective space defined by a homogeneous polynomial of degree two.)

7. Grassmannians. Fix a vector space $V$ of finite dimension $n$, over an arbitrary ground field $k$. Let $\mathbb{G}_d(V)$ denote the set of all $d$-dimensional subspaces of $V$. Thus $\mathbb{G}_1(V) = \mathbb{P}(V)$.

a). Explain how the set of lines in $\mathbb{P}(V)$ is naturally identified with the set $\mathbb{G}_2(V)$, and that in general, the linear spaces in $\mathbb{P}(V)$ of dimension $d$ are the points in $\mathbb{G}_{d+1}(V)$.

b). Fix a basis for $V$, thereby identifying $V$ with $k^n$. Use this to represent any $d$-dimensional subspace $W$ as a $d \times n$ matrix of scalars, of full rank $d$.

c). Prove that two rank $d$ matrices $A_1$ and $A_2$ of size $d \times n$ determine the same subspace $W$ if and only if there exists an element $g \in GL(d,k)$ such that $A_1 = gA_2$. Conclude that the set $\mathbb{G}_d(V)$ can be identified (as a set) with the quotient of the set of all full rank $d \times n$ matrices by the natural action of $GL(d)$ on the left.\(^2\)

d). Show that $\mathbb{G}_d(V)$ is covered by $\binom{n}{d}$ sets $U_I$, each identified with affine space $\mathbb{A}^{d(n-d)}$ in a natural way. In the case $d = 1$, your cover should specialize to the standard affine cover of $\mathbb{P}(V)$.

e). Let $U_1$ and $U_2$ be two of the sets identified in (d), and let $V_i$ be the subset of $\mathbb{A}^{d(n-d)}$ corresponding to $U_1 \cap U_2$ under the identifications $\phi_i : U_i \to \mathbb{A}^{d(n-d)}$. Prove that $V_i$ is open in $\mathbb{A}^{d(n-d)}$, and explicitly describe the chart change map $\phi_2 \circ \phi_1^{-1}$.

f). Explain how $\mathbb{G}_d(V)$ is an abstract variety\(^3\) over any field $k$, a smooth manifold over $\mathbb{R}$, and a complex manifold over $\mathbb{C}$ . What is its dimension?

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\(^2\)If the ground field is, say $\mathbb{R}$, this endows $\mathbb{G}_d(V)$ with a natural structure as a topological space, which agrees with the Hausdorff topology on $\mathbb{P}(V)$ in the case $d = 1$.

\(^3\)Even better, on the next problem set, we will see that $\mathbb{G}_d(V)$ has an embedding into projective space which gives it the structure of a projective variety.