Math 631: Problem Set 5

Due Friday October 10 , 2008

1. Closed sets of $\mathbf{P}^n \times \mathbf{P}^m$. Recall that the closed sets in $\mathbf{P}^n \times \mathbf{P}^m$ are given by bihomogeneous polynomials (see Shaf 5.1).

a). Let X be a hyperplane section of the Segre variety Σ_{nm} in $\mathbf{P}^{(m+1)(n+1)-1}$, that is, the intersection of a hyperplane in $\mathbf{P}^{(m+1)(n+1)-1}$ with the image of $\mathbf{P}^n \times \mathbf{P}^m$ under the Segre map σ_{nm}. Explicitly describe the bihomogeneous polynomials that define the corresponding closed subset of $\mathbf{P}^n \times \mathbf{P}^m$.[1]

b). Let $X = \mathbb{V}(x^2y^3uv + x^5v^2 + y^4xu^2)$ be a closed set in $\mathbf{P}^1 \times \mathbf{P}^1$ (where homogeneous coordinates are $x : y$ on first copy and $u : v$ on second copy of $\mathbf{P}^1$). Find defining equations for the image of X under the (restriction of) the segre embedding $\sigma_{1,1}$ in $\mathbf{P}^3$, with coordinates $z_{00}, z_{01}, z_{10}, z_{11}$.

2. Resultant. Let F and G be two homogenous polynomials in $k[U, V]$, of degrees m and n respectively. Let $\mathbf{Sym}^{n+m-1}(k^2)^*$ denote the vector space of homogeneous polynomials in $k[U, V]$ of degree $m + n - 1$.

a). Show that F and G have a common factor if and only if the subvector spaces V_F and V_G of $\mathbf{Sym}^{n+m-1}(k^2)^*$ of polynomials divisible by F (respectively G) meet non-trivially.

b). Show that F and G have a common factor if and only if the polynomials

$$U^{n-1}F, U^{n-2}VF, \ldots, V^{n-1}F, U^{m-1}G, U^{m-2}VG, \ldots, V^{m-1}G$$

are linearly dependent.

c). Show that F and G have a common factor if and only if the determinant of a certain $(m+n) \times (m+n)$ matrix formed from the coefficients of F and G is zero. This matrix is called the *resultant* of F and G.

d). Assume $k = \bar{k}$. Let $\mathbf{P}(\mathbf{Sym}^m(k^2)^*)$ and $\mathbf{P}(\mathbf{Sym}^n(k^2)^*)$ be the projective spaces of all homogeneous polynomials in U and V of degree m and n respectively. Consider the subset $\Gamma = \{(F, G) \mid \mathbb{V}(F) \cap \mathbb{V}(G) \neq \emptyset$ in $\mathbf{P}^1\}$ of $\mathbf{P}(\mathbf{Sym}^m(k^2)^*) \times \mathbf{P}(\mathbf{Sym}^n(k^2)^*)$. Prove the subset Γ is a non-empty Zariski closed set.

3. Blowing up. Let $X \subset \mathbb{A}^2 \times \mathbf{P}^1$ be the set of pairs (p, ℓ), where $p \in \mathbb{A}^2$ and ℓ is a line through the origin in $\mathbb{A}^2$ containing p.

a). Prove that X is a closed set in $\mathbb{A}^2 \times \mathbf{P}^1$. Find explicit defining equations.

b). Consider the natural map $\pi : X \to \mathbb{A}^2$ given by projection onto the first coordinate. Show that this map is a surjective regular map. For each point p of $\mathbb{A}^2$, describe the preimage set $\{\pi^{-1}(p)\}$ (in particular, what are its defining equations? what well-known variety is it isomorphic to? its dimension?). Is π finite?

c). Consider the natural map $\eta : X \to \mathbf{P}^1$ given by projection onto the second factor. Show that it is a surjective regular map. Describe the preimage of each point $p \in \mathbf{P}^1$. Is this map finite? This map defines what is called the *tautological line bundle* on $\mathbf{P}^1$. Without going into technicalities about the definition of line bundles, why is this name justified?

[1]Hint: Let z_{ij} denote the homogeneous coordinates on $\mathbf{P}^{(m+1)(n+1)-1}$.

4. Family of Degenerate Conics. For this problem assume the field does not have characteristic 2.

a). Show that the subset of degenerate conics in $\mathbf{P}^2 = \mathbf{P}(V)$ (those that are a union of two lines or a "double line") forms a proper projective subvariety of $\mathbf{P}^5 = \mathbf{P}(Sym^2(V^*))$ isomorphic to a certain projection of the Segre image of $\mathbf{P}^2 \times \mathbf{P}^2$ in $\mathbf{P}^8$ to $\mathbf{P}^5$. What is the dimension of the subvariety of degenerate conics?[2]

b). Show that the subset of "double lines" forms a proper closed subset of the space of all conics isomorphic to the Veronese surface in $\mathbf{P}^5$.

5. A typical finite map. Let V be an irreducible hypersurface in $\mathbf{P}^n$ over an algebraically closed field of characteristic zero. Suppose that the irreducible polynomial defining V can be written in the form $x_n^d + a_1 x_n^{d-1} + \ldots + a_d$, where the a_i are homogeneous of degree i in the variables $x_0, \ldots, x_{n-1}$. Consider the projection $\pi : V \longrightarrow \mathbf{P}^{n-1}$ from $p = [0 : 0 : \ldots : 0 : 1]$ to $H = \mathbb{V}(x_n)$.

a). Show that π is surjective. Describe the preimage of each point of $\mathbf{P}^{n-1}$. The points of $\mathbf{P}^{n-1}$ have preimages of the same cardinality, if we count them with multiplicities, this cardinality is called the *degree* of the map. How do we assign the multiplicities? What is the degree of π?

b). The points whose pre-image fails to have precisely degree π distinct points are called *ramification points.* Prove that the set of ramification points (the ramification locus) is a proper Zariski closed subset of $\mathbf{P}^{n-1}$, in fact, a hypersurface in $\mathbf{P}^{n-1}$. (Hint: Remember a criterion from Galois theory for when a polynomial in a single variable has a repeated root.)

c). Prove that π is finite, directly from the definition of finiteness (that is, given by integral extensions of the coordinate rings locally.)

6. The Plücker Embedding. a). Let $N = \binom{n}{d}$. Show that there is a well defined map

$$\phi : \mathbb{G}_d(V) \to \mathbf{P}^{N-1} = \mathbf{P}(\wedge^d V)$$

which sends any d-dimensional subspace W to $\wedge^d W \subset \wedge^d V$. Show also that ϕ is one-to-one onto its image, and that the image is precisely the set of (one dimensional) subspaces of $\wedge^d V$ spanned by indecomposable vectors in $\mathbf{P}(\wedge^d V)$ (that is, vectors of the form $v_1 \wedge v_2 \ldots \wedge v_d$.)

b). Show that ϕ can be expressed explicitly in coordinates as taking a $d \times n$ matrix representing a d-dimensional space W to an N-tuple consisting of its maximal minors.

c). Show that a point $[\omega]$ in $\mathbf{P}(\wedge^d(V))$ (represented by a vector $\omega \in \wedge^d(V)$) lies in the image of ϕ if and only if the map

$$\wedge\omega : V \to \wedge^{d+1} V$$

sending v to $v \wedge \omega$ is rank $n - d$. (Hint: First show that a vector $\omega \in \wedge^d V$ is indecomposable if only if the space of vectors that "divide" it in the exterior algebra has dimension d.)

d.) Show that ϕ identifies $\mathbb{G}_d(V)$ with a Zariski closed subset of $\mathbf{P}(\wedge^d V)$. Thus Grassmannians are projective varieties![3]

e). Recall that on Problem Set 4, you found a nice affine cover of the Grassmannian. Show that this cover corresponds to the standard affine open cover under the identification of $\mathbb{G}_d(V)$ with a projective variety in $\mathbf{P}(\wedge^d V)$ via the embedding ϕ. Is $\mathbb{G}_d(V)$ irreducible? What is its dimension?

[2] Hint: $\mathbf{P}^2 \times \mathbf{P}^2$ is really $\mathbf{P}(V^*) \times \mathbf{P}(V^*)$.

[3] If you used (c) to prove (d), the equations you get do *not* generate the full ideal of all homogeneous polynomials equations vanishing on $\mathbb{G}_d(V)$. It turns out that this ideal is generated by *quadratic polynomials,* called Plücker relations, described on p 42 of Shafarevich.