Math 631: Problem Set 6

Due Friday October 17 , 2008

1. Find the dimensions of the following varieties:

a). The affine cone over a projective variety of dimension d.

b). The variety of m by n matrices of rank t,

c). The twisted cubic in $\mathbf{P}^3$.

d). The product of two irreducible varieties of dimensions d and d'.

2. More on Dimension. a). Fix a point Q and any points $P_1, \dots, P_t$ in $\mathbf{P}^n$. For any d, Show that there is a hypersurface of degree d in $\mathbf{P}^n$ that passes through Q but none of the P_i. What more can you say about the set of all such hypersurfaces with this property in $\mathbf{P}(Sym^d(V^*))$?

b). Show that the dimension of an irreducible quasi-projective variety V is equal to the length of the longest chain of closed irreducible subvarieties of V. (The projective case was done in class.)

c). Define the dimension of a variety V at a point x to be the length of the longest chain of closed irreducible subvarieties of V containing x. Denote this quanitity by $\dim_x(V)$. If V is irreducible, prove that $\dim_x(V)$ is the same for all points $x \in V$.

d). Show that for any point x on an irreducible variety V of dimension d, there exist regular functions $f_1, \dots, f_d$ in some neighborhood U of x such that, on U, the common zero set of the f_i is precisely y.

3. Degree of a Finite Map. Let $\phi : X \to Y$ be a finite morphism of irreducible varieties.

a). Explain how ϕ induces an identification of $k(Y)$ with a subfield of $k(X)$. We define the *degree* of ϕ to be the degree of this field extension.

b). Verify that the map you studied in problem 5 on homework set 5 has degree d in this sense.

c). Compute the degree of $\mathbf{P}^1 \to \mathbf{P}^1$ sending $[s : t] \mapsto [s^d : t^d]$. Describe the fibers.

4. Degree of a Hypersurface. Let X be an irreducible hypersurface in $\mathbf{P}^n$ of degree d, meaning that the irreducible homogenous polynomial of X has degree d.

a). Show that every line in $\mathbf{P}^n$ intersects X in exactly d points (counting multiplicity), unless it lies on X.

b). Show that X admits a finite cover of $\mathbf{P}^{n-1}$ of degree d in the sense of Problem 3.

5. Another interesting finite map. Fix a projective variety $V \in \mathbf{P}^n$. Let $F_0, \dots, F_m$ be homogeneous polynomials of degree d in $k[x_0, \dots, x_n]$ that do not simultaneously vanish at any point of V. Consider the map $\phi : V \longrightarrow \mathbf{P}^m$ sending a point x to $[F_0(x) : \dots : F_m(x)]$.

a). Prove that ϕ is finite onto its image by reducing the case where $d = 1$ and using the fact that projections from linear spaces are finite.

b). Prove ϕ is finite by directly thinking about the preimage of each point and using (without proof) the theorem that for projective varieties, a dominant morphism is finite if and only if it has finite fibers.

c) Can you give any bounds on the number of points in the pre-image of a typical point?

6. Diagonal Maps. Consider the diagonal mapping $\Delta : \mathbf{P}^n \subset \mathbf{P}^n \times \mathbf{P}^n$ sending each x to the pair (x, x).

a). Prove that Δ defines an isomorphism between $\mathbf{P}^n$ and some closed set of $\mathbf{P}^n \times \mathbf{P}^n$, and find an explicit set of bihomogeneous generators for the image.

b). For any quasiprojective V, show that the diagonal $\Delta_V = \{(x, x) | x \in V\}$ is closed in $V \times V$.

c). Prove that the intersection of two affine open subsets of a quasi-projective variety is affine.[1]

7. Affine Schemes—the starting point of Math 632. Let R be any commutative ring. Let Spec(R) denote the set of all prime ideals of R, and mSpec(R) denote the set of maximal ideals of R. (These are called the prime spectrum and the maximal spectrum of R, respectively).

a). Show that Spec R is a topological space with closed sets $\mathbb{V}(I) = \{P \mid P \supset I\}$, where I is any ideal of R, and that the sets $U_f = \{P \mid f \notin P\}$ form a basis. This topological space is called an affine scheme.[2]

b). Let R be the coordinate ring of an affine variety X. Show that X is homeomorphic to mSpec R (with the subspace topology).

c). Show that every ring homomorphism $\phi : R \to S$ induces a continuous map Spec $S \to$ Spec R sending P to $\phi^{-1}(P)$. If R and S are both finitely generated reduced k-algebras[3] and ϕ is a k-algebra map, show that ϕ also induces a map of the maximal spectra, which recovers the morphism of the corresponding varieties under the homeomorphism of (b).

d). For a ring map $\phi : R \to S$, show that the fiber over $P \in$ Spec R under the induced map on Spectra is homeomorphic to Spec$(R_P/PR_P \otimes_R S)$.

e). If $\phi : R \to S$ is an integral extension of domains, show that the induced map on the maximal spectra is finite-to-one. If R and S happen to be finitely generated algebras over an algebraically closed field, what is the corresponding geometric statement?

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[1]This is one property of quasi-projective varieties that can fail for abstract varieties as we have defined them. Usually, one adds the property that the diagonal is closed (such a variety is said to be "separated") to the definition of an abstract variety to rule out such pathological behavior. For example, as we defined abstract variety, one can put the structure of an abstract variety on the affine line with a "doubled origin", a slight variant on the classic example of a non-Hausdorff space from Math 590. Indeed, separatedness for algebraic varieties is the analog of "Hausdorff" in topology.

[2]Really, a scheme is a ringed space, just like a variety is a ringed space, so we should put a sheaf of rings on Spec R. This is not hard, but maybe this problem set is long enough. On U_f, the sheaf takes the value $R[\frac{1}{f}]$, and the restriction maps are localizations.

[3]where k is algebraically closed