Math 631: Problem Set 7

Due Wednesday October 29, 2008

1. Smooth Quadrics. Prove that a quadric hypersurface in $\mathbf{P}^n$ is smooth if and only if the associated quadratic form is non-singular.

2. Maps on Tangent Spaces. Say that V and W are closed subvarieties of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively, and let $F : V \longrightarrow W$ be a regular map. Find and prove an explicit description of the induced linear map on tangent spaces $d_pF : T_pV \longrightarrow T_{F(p)}W$ at a point $p \in V$ in terms of the Jabobian matrix of the coordinate functions describing F.

3. Let V be the affine variety in $\mathbb{A}^3$ consisting of the union of the three coordinate axes, and W be the affine variety in $\mathbb{A}^2$ consisting of the union of the x-axis, y-axis, and line $y = x$. Prove or disprove that V and W are isomorphic.

4. Projective Tangent Space. Let p be a point on an irreducible projective algebraic variety $V \subset \mathbf{P}^n$. Show that the following definitions of the "projective tangent space" to V at p are equivalent:

a). The set of all points lying on lines in $\mathbf{P}^n$ tangent to V at p.

b). The closure in $\mathbf{P}^n$ of the tangent space to the affine variety $V \cap U_i$ at p, where U_i is any standard affine chart containing p.

c). The projective linear space corresponding to the subspace of k^{n+1} which is the kernel of the $r \times (n+1)$ scalar matrix obtained by evaluating the Jacobian matrix $\frac{\partial g_i}{\partial x_j}$ at $\tilde{P}$, where $g_1, \dots, g_r$ are homogeneous generators for the radical ideal of V and $\tilde{P}$ is any $n + 1$-tuple representing p.

d). The projective space $\mathbf{P}(T_{\tilde{P}}\tilde{V})$, where $\tilde{V} \subset k^{n+1}$ is the affine cone over V in k^{n+1} and $\tilde{P}$ is any point in k^{n+1} representing the point p in $\mathbf{P}^n$.

5. Tangent Fiber Space. Let V be a closed irreducible subvariety of $\mathbf{P}^n$. Prove that the "tangent fiber space"

$$TV = \{(p, q) | q \in T_pV\} \subset V \times \mathbf{P}^n,$$

where T_pV here is the projective tangent space to V at p is a Zariski closed set of $V \times \mathbf{P}^n$. Show that TV is irreducible when V is smooth and compute its dimension.

6. The family of hyperplane sections. Let $X \subset \mathbf{P}(V)$ be an irreducible projective variety, not a point. Consider the set

$$\mathbb{H} = \{(p, H) \,|\, p \in H\} \subset X \times \mathbf{P}(V^*),$$

where $\mathbf{P}(V^*)$ is the set of hyperplanes in $\mathbf{P}(V)$.

a). Prove that the projection $\pi : \mathbb{H} \longrightarrow \mathbf{P}^n(V^*)$ (sending $(p, H) \mapsto H$) is surjective. Give a nice geometric interpretation of the fiber $\pi^{-1}(H)$. What interesting variety is it naturally isomorphic to?

b). What is the dimension of the generic fiber of π? Find a nice geometric condition on X guaranteeing that all fibers have this dimension.

c). Prove that $\mathbb{H}$ is a closed *irreducible* subvariety of $X \times \mathbf{P}^n$. What is its dimension?

7. Embedding Dimension. By definition, the embedding dimension of a point p on a variety V is the dimension of the Zariski tangent space of V at p.

a). Prove the name is apt: if a point admits some open neighborhood isomorphic to a locally closed subset of $\mathbb{A}^n$, then its embedding dimension is at most n.

b). Compute the tangent space to C at the origin, where C is the image of the affine line under the map $t \mapsto (t^n, t^{n+1}, \dots, t^{2n-1})$ to $\mathbb{A}^n$.

c). Show that for every N, there is a curve which can not be embedded in $\mathbb{A}^N$.[1]

[1] The curve you construct will necessarily be singular. It is a theorem that every smooth curve embeds in $\mathbf{P}^3$.