

Math 631: Problem Set 8

Due Wednesday November 5, 2008

1. A family of elliptic curves. For $\lambda \in k$, let C_λ denote the plane projective cubic curve defined by $x^3 + y^3 + z^3 + \lambda(x + y + z)^3$. Construct a family¹ of varieties parametrized by the affine line whose members are exactly the C_λ . Explicitly describe the locus of points in your parameter space \mathbb{A}^1 parametrizing those C_λ that are smooth.

2. Singular set of Projective Varieties. Let $V \subset \mathbf{P}^n$ be a projective variety and let $\tilde{V} \subset k^{n+1}$ be the affine cone over it.

a). Prove that V is smooth if and only if \tilde{V} has (at worst) an isolated singular point at the origin.

b). Prove that \tilde{V} is always singular at the origin unless V is a linear variety in \mathbf{P}^n .

c). If the homogeneous ideal of V is generated by homogeneous polynomials F_1, F_2, \dots, F_m and V has pure codimension c , show that $\text{Sing } V = V \cap \mathbb{V}(c \times c \text{ minors } \frac{\partial F_i}{\partial x_j})$.

3. Universal Hypersurface. Consider the variety \mathcal{X} defined by the bihomogeneous polynomial $\sum a_I x^I$ in $\mathbf{P}(\text{Sym}^d(V^*)) \times \mathbf{P}(V)$, where x^I denotes $x_0^{i_1} \dots x_n^{i_n}$ for x_0, \dots, x_n coordinates for $\mathbf{P}(V)$ and a_I the corresponding coordinates on $\mathbf{P}(\text{Sym}^d(V^*))$.

a). Explain how this allows us to think of the set of hypersurfaces in $\mathbf{P}(V)$ as a family in the technical sense of the word, parametrized by $\mathbf{P}(\text{Sym}^d(V^*))$.

b). Prove that \mathcal{X} is an irreducible projective variety by considering one of the two obvious projections.

c). Consider the subset $\Sigma \subset \mathcal{X}$ consisting of pairs $(H, p) \in \mathbf{P}(\text{Sym}^d(V^*)) \times \mathbf{P}(V)$ where p is a *singular* point on the hypersurface in $\mathbf{P}(V)$ corresponding to H . Prove that Σ is a proper closed subset of \mathcal{X} by giving some explicit defining equations.

d). What is the geometric significance of the image of Σ under the projection to $\mathbf{P}(\text{Sym}^d(V^*))$? Can the image of Σ be all of $\mathbf{P}(\text{Sym}^d(V^*))$?

4. Bertini's Theorem. a). Let $V = \mathbb{V}(F)$ be a smooth irreducible hypersurface in \mathbf{P}^n , not contained in any hyperplane \mathbf{P}^{n-1} . Prove that a hyperplane section $V \cap H$ is smooth at p if and only if the hyperplane H is not tangent to V at p .

b). Prove that the set of smooth hyperplane sections of V is parametrized by a non-empty Zariski open subset of the variety of hyperplanes in \mathbf{P}^n . (This is called Bertini's theorem, and is true even if V is not a hypersurface.)

¹in the technical sense of a surjective morphism of varieties

5. Gauss Map. Let X be a smooth irreducible projective variety of dimension d in $\mathbf{P}(V) = \mathbf{P}^n$. Define a map $\rho : X \rightarrow \mathbb{G}_{d+1}(V)$ sending the point p to the projective tangent space to X at p .

- a). Explain why there is a natural identification of $\mathbb{G}_{d+1}(V)$ with $\mathbb{G}_{n-d}(V^*)$.
- b). Prove that ρ is a morphism of varieties. What is going on when two points have the same image under ρ ?
- c). Let $X = \mathbb{V}(x^d + y^d + z^d)$ in \mathbf{P}^2 . Compute ρ explicitly. What is the dimension of the image²? Compute the image explicitly in the cases where $d = 1, 2$, and 3 .

6. The Tangent Cone. a). Read page 95 of Shafarevich (section II 1.5). Accept the fact³ that the tangent cone of any variety at p has the same dimension as V at p . Show that if p is a smooth point of V , then the tangent cone to V at p agrees with the tangent space to V at p .

- b). Prove that if V is a plane curve, then the tangent cone to V is a union of lines, and find the tangent cones at the origin for each of the plane curves below. Sketch each curve and its tangent cone.

$$y^2 - x^2 - x^3 = 0, \quad xy - y^4 = 0.$$

- d). How do you expect the (complex points of the) curve defined by the equation $x^3y - y^3x - xy^{17} + \pi y^{203}$ to look in a tiny ϵ -neighborhood of the origin (using the Euclidean topology)? What about $x^2y + xy^2 - x^4 - y^4 = 0$?
- e). Fix a finite set of scalars $\{m_1, \dots, m_k\}$. Show that there exists a plane curve C passing through a point p which has k branches passing through p with slopes m_1, \dots, m_k , respectively.
- f). Find the tangent cone to the curve $\mathbb{V}(x^3 - y^2)$ at the origin. Why would it be better to think of this as a “scheme”? What scheme should it be?

²The image is called the dual curve.

³Prove it for extra credit.