Math 631: Problem Set 9

Due Friday November 14, 2008

1. Blowup along Principle ideals. Recall that the blowup of an affine variety X along an ideal I generated by regular functions $f_0, \ldots, f_t$ is the graph of the rational map $X \dashrightarrow \mathbf{P}^t$ sending $x \mapsto [f_0(x) : \ldots : f_t)]$.

Show that the blowup of any irreducible affine variety along a principle ideal is an isomorphism.

2. Bad Blowing up. Show that the blowup X of $\mathbb{A}^2$ along the ideal (x^2, y^3) is not smooth. Find defining equations for the singular locus of X as a subvariety of $\mathbb{A}^2 \times \mathbf{P}^1$. (Thus bad blowing up can make the singularities of a variety get *worse* !)

3. Blowing up a line. a). Let X be the blow-up of $\mathbb{A}^3$ along any line L. Let $\pi : X \longrightarrow \mathbb{A}^3$ denote the blowing up morphism along L. Find defining equations for X as a subvariety of $\mathbb{A}^3 \times \mathbf{P}^1$, and prove that X is smooth.[1]

b). The *exceptional set* of a birational morphism $\phi : X \to Y$ is the closed subset of X at which ϕ is not an isomorphism. Describe the exceptional set of the blowup π (from (a)) both geometrically and algebraically (by giving its defining equations.) What is its dimension?

c). Fix a point p on L. Consider a line ℓ in $\mathbb{A}^3$ through p other than L. Describe, both algebraically and geometrically, the point (or points) in X in the fiber over p which lie on the proper transform of the line ℓ. (The proper transform of ℓ is the closure of $\pi^{-1}(\ell \setminus L \cap \ell)$ in X.)

4. Nodal Curve Let $C = \mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{A}^2$. Let $\pi : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ be the blowup of the origin in $\mathbb{A}^2$.

a). Show that $\pi^{-1}(C)$ is a reducible curve with two components, one isomorphic to $\mathbf{P}^1$.

b). Find the intersection points of the two components, expressing your answer in terms of coordinates on an appropriate affine patch of $\tilde{\mathbb{A}}^2$. Explain the geometric meaning of these intersection points.

c). The component of $\pi^{-1}(C)$ other than $\mathbf{P}^1$ is called the proper transform of C. Show that the proper transform of C is equal to the (Zariski) closure of $\pi^{-1}(C - \{(0,0)\})$.

d). Show that the proper transform $\tilde{C}$ of C is a non-singular variety, which is birationally equivalent to C. Describe the map $\pi : \tilde{C} \to C$. How does this compare to $\pi^{-1}(C) \to C$?

e). Illustrate all this with a picture, choosing a suitable affine patch of $\mathbb{A}^2 \times \mathbf{P}^1$.

[1]Please do not torture yourself with poor choice of coordinates.

5. Transverse intersection. a). Let $\{W_i\}_{i=1}^t$ be a finite set of subspaces of a finite dimensional vector space V. Show that $\operatorname{codim}(\bigcap_{i=1}^t W_i) \leq \sum_{i+1}^t \operatorname{codim} W_i$. We say that the W_i intersect *transversely* if equality holds. *Draw some pictures and ponder the meaning! For example: if $t > n$, what happens? if $t = 2$, what typically happens? what else can you say?*

b). Show that if $W \subset V$ is an inclusion of varieties, then there is an injective linear map $T_pW \subset T_pV$ for any $p \in W$.

c). Let W_i be a finite collection of closed subvarieties of a smooth variety V, all containing a point P. Show that $\operatorname{codim}(\bigcap_{i=1}^t T_pW_i) \leq \sum_{i+1}^t \operatorname{codim} W_i$. We say that the W_i intersect tranversely if equality holds. Show that subvarieties of a smooth variety V *intersect transversely at* P if and only if each is smooth at P and their tangent spaces at P intersect transversely as subspaces of T_PV.

d). Draw some pictures of varieties in three-space that do and do not intersect transversely.

d). Let $u_1, \ldots, u_d$ be parameters at a smooth point P of a variety V, and let W_i be closed subsets in a neighborhood of P defined by the vanishing of disjoint subsets of the u_j. Show that the W_i intersect transversely at P.

6. Multiple Points. Let X be an irreducible hypersurface. The *multiplicity of* a point P on X is the unique integer r such that a local defining equation for X at P is in $m_P^r \setminus m_P^{r+1}$, where $m_P \subset \mathcal{O}_{\mathbf{P}^n,P}$ is the maximal ideal of germs of regular functions on $\mathbf{P}^n$ vanishing at P.

a). Prove that P is a smooth point of X if and only if it is a point of multiplicity 1.

b). Show that a line through a multiplicity r point P must intersect X at P with multiplicity $\geq r$.

c). Now assume the hypersurface X has degree d. Show all points on X have multiplicity $\leq d$.

d). Show that X has a multiplicity d point P if and only if X is a cone with vertex at P (meaning that every line through P meets X only at P or lies on X).

e). Show that if X has a multiplicity $d-1$ point and is not a cone with vertex at P, then X is birationally equivalent to $\mathbf{P}^{n-1}$.[2]

[2] This shows that there are rational hypersurfaces of arbitrarily large degree. However, a smooth hypersurface of degree $d > N$ in $\mathbf{P}^n$ is never rational. Whether or not a smooth hypersurface of degree $3 \leq d \leq n$ can be rational is mostly an open question.