Math 217: The Determinant of a linear transformation
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Inquiry: What is the geometric meaning of the determinant?

Definition: Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be a \( 2 \times 2 \) matrix. The determinant of \( A \) is the scalar \( ad - bc \).

A. Let \( \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation which stretches each vector by 2 in the horizontal direction and by 3 in the vertical direction.

1. Draw a picture of the source \( \mathbb{R}^2 \), showing the vectors \( \vec{e}_1 \) and \( \vec{e}_2 \). Draw a separate picture of the target \( \mathbb{R}^2 \), showing their images under \( T \). Show also the vectors \( \vec{e}_1 + \vec{e}_2 \) and \( T(\vec{e}_1 + \vec{e}_2) \).

2. Write an algebraic formula for \( T(\begin{bmatrix} x \\ y \end{bmatrix}) \).

Solution note: \( T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x \\ 3y \end{bmatrix} \).

3. Find a matrix \( A \) such that for every vector \( \begin{bmatrix} x \\ y \end{bmatrix} \), we have \( T(\begin{bmatrix} x \\ y \end{bmatrix}) = A \begin{bmatrix} x \\ y \end{bmatrix} \). Is \( A \) unique? What are its columns in terms of \( \vec{e}_1 \) and \( \vec{e}_2 \)?

Solution note: \( A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \). It is unique, as its columns are determined exactly: the first column is \( T(\vec{e}_1) \) and the second column is \( T(\vec{e}_2) \).

4. Shade the set \( R = \{ c_1 \vec{e}_1 + c_2 \vec{e}_2 \mid 0 \leq c_i \leq 1 \} \) in the source. Shade the image \( T(R) \) of this set in the target. Write the image in set-builder notation.

Solution note: The image \( T(R) \) is a rectangle of height 3 and width 2, squared up against the \( x \) and \( y \) axis in the first quadrant. \( T(R) = \{ c_1 T(\vec{e}_1) + c_2 T(\vec{e}_2) \mid 0 \leq c_i \leq 1 \} \) or alternatively, \( T(R) = \{ \begin{bmatrix} 2c_1 \\ 3c_2 \end{bmatrix} \mid 0 \leq c_i \leq 1 \} \).

5. Compute the area of \( R \) and of \( T(R) \).

Solution note: \( R \) has area one and \( T(R) \) has area 6.

6. Compute the determinant of the matrix of \( A \). What do you notice?

Solution note: The determinant is 6, same as the area expansion factor!

B. Let \( \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation given by multiplication by the matrix \( \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \).

1. Draw a picture (in the target \( \mathbb{R}^2 \)) of the image of the unit square under \( S \). Label the vertices.

2. Compute the area of the parallelogram in (1) (geometrically).
3. Compare to the determinant of the matrix of $S$.

4. Suppose $Q$ is a triangle of area 5, what shape is $S(Q)$ and what is its area?

5. Suppose $C$ is a circle of area $A$, what is the area of the distorted circle $S(C)$?

6. Suppose $B$ is a random blob of a shape with area 22 square units. What is the area of the distorted blob $S(B)$?

**Solution note:** The unit square is stretched and pulled into a parallelogram with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Its area is 4, same as the determinant of the matrix. Likewise, triangles, circles, and any shape at all get pulled/stretched/distorted, but the area of the new shape will always be 4 times the area of the old!

C. Let $\mathbb{R}^2 \xrightarrow{\beta} \mathbb{R}^2$ be given by multiplication by $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$. What does it do to the unit square? What is its determinant? Does this fit in with your observations in A and B?

**Solution note:** Here, the unit square is squashed onto a line segment whose endpoints are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$. The “area” of the line segment is zero, same as the determinant of the matrix!

D. Let $\mathbb{R}^2 \xrightarrow{\phi} \mathbb{R}^2$ be the map that swaps $x$ and $y$ coordinates. Is this linear? If so, what is the corresponding matrix? What does it do to the unit square? What is its determinant? Does this fit in with your observations in A, B, and C?

**Solution note:** The formula for $\phi$ is $\phi(\begin{bmatrix} x \\ y \end{bmatrix})$. Using this formula we can easily verify that $\phi(\vec{v} + \vec{w}) = \phi(\vec{v}) + \phi(\vec{w})$ and that $\phi(\alpha \vec{v}) = \alpha \phi(\vec{v})$ for all vectors $\vec{v}, \vec{w} \in \mathbb{R}^2$ and all scalars $\alpha$. So $\phi$ is linear. Its matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It maps the unit square right back onto itself, by flipping it over line $y = x$ (the diagonal of the square). So the area of $T(R)$ is the same as the area of $R$ (which is 1), although the determinant of the matrix is $-1$ not 1.

E. Conjecture a geometric meaning of the determinant. Does your conjecture make sense when the determinant is zero? negative? You will prove your conjecture and higher dimensional versions of it in Chapter 6.

**Solution note:** Conjecture: A linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ distorts shapes but for any shape $\Omega$ of area $\alpha$, the area of $T(\Omega)$ will be $|\text{det}A|\alpha$. This is in fact a true theorem, and it will work in higher dimension too (using volume for $\mathbb{R}^3$ and a higher analog of volume in $\mathbb{R}^n$). You will eventually be able to prove this.