Fall 2015 Math 217 Section 5
Professor Karen Smith
Quiz 5

A. Define basis of a vector space V.

> *Solution note:* A basis is a set of linearly independent vectors that span V.

B. Suppose W is a subspace of $\mathbb{R}^n$ with basis $\vec{v}_1, ..., \vec{v}_d$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Assume that $T(\vec{v}_i) = 0$ for all $i = 1, \dots, d$.

(1) Prove that $W \subset \ker T$.
(2) Prove that if $\dim \operatorname{im}(T) = n - d$, then $\ker(T) = W$.

> *Solution note:* (1). Take $\vec{w} \in W$. We need to show $T(\vec{w}) = 0$. Since the $\vec{v}_1, ..., \vec{v}_d$ span W, we can write $\vec{w} = a_1\vec{v}_1 + \cdots + a_d\vec{v}_d$. Apply T. We have $T(\vec{w}) = T(a_1\vec{v}_1 + \cdots + a_d\vec{v}_d) = a_1T(\vec{v}_1) + \cdots + a_dT(\vec{v}_d)$, by the linearity of T. Since all the $T(\vec{v}_i) = 0$, it follows that $T(\vec{w}) = 0$.
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> (2) By rank nullity, the hypothesis tells us that the dimension of the kernel is $n - (n - d) = d$. So $W \subset \ker T$, and both have dimension d. By a result proved in the homework, $W = \ker T$. Alternatively, we can say that the vectors $\vec{v}_1, ..., \vec{v}_d$ are a linearly independent set of d vectors in $\ker T$, which has dimension d, so by Theorem 3.3.4 in the book, they must be a basis.

C. Suppose that a series of elementary row operations on B results in the matrix $\hat{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$. (Note that $\hat{B}$ may not be $\operatorname{rref}(B)$.) If B is not invertible, what are the possible values of p, q, and r? Justify your answer.

> *Solution note:* We have that $\operatorname{rref}(B) = \operatorname{rref}(\hat{B}) = I_3$ if and only if $p \neq 0$ and $r \neq 0$. Therefore, B is invertible if and only if p and r are both nonzero. If B is not invertible, then either $p = 0$ and q, r are arbitrary or $r = 0$ and p, r are arbitrary.