Math 217: Quiz 5B

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The *Fibonacci sequence* $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is both mathematically interesting and has a habit of showing up in nature. It is defined recursively, via the rules $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Of course, the computation of large Fibonacci numbers is a massive pain to do directly. Here, we use matrices and show how wisely choosing bases can give a shortcut.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

1. Compute A^2, A^3, A^4. What do you see?

2. Prove that $A^n \vec{e}_1 = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ for all $n \geq 1$.

3. What is the 51-st Fibonacci number in terms of the matrix A? Can you make a computer find it?

4. Suppose we could find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ for $\mathbb{R}^2$ such that $B = [T_A]_{\mathfrak{B}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Write a matrix equation expressing the relationship between A and B (your answer should involve the $\vec{v}_i$). We say that this basis *diagonalizes* the linear transformation T_A given by (left) multiplication by A.

5. Express A^n in terms of B and some other matrices constructed from the $\vec{v}_i$. How does this make computing F_{51} easier?

   Solution note: Let $S = [\vec{v}_1 \quad \vec{v}_2]$. This is a 2×2 matrix, the change of coordinates matrix from $\mathfrak{B}$ to the standard basis. By a theorem in the book, $B = S^{-1}AS$, so $A = SBS^{-1}$. So $A^n = (SBS^{-1})^n$. Expanding this out, the interior factors of S and S^{-1} cancel out, so $A^n = SB^nS^{-1}$.

6. Suppose you could find two linearly independent vectors $\{\vec{v}_1, \vec{v}_2\}$ such that $A\vec{v}_i$ is parallel to $\vec{v}_i$, that is, such that $A\vec{v}_i = \lambda_i \vec{v}_i$. How does this help the diagonalizing we want from (4).

7. Find two values of λ such that $A \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ for some vector $\begin{bmatrix} x \\ y \end{bmatrix}$. Set λ_1 to be the larger number and λ_2 the smaller. [Hint: Note that if $A\vec{v} = \lambda\vec{v}$, then the same is true for any scalar multiple of $\vec{v}$. So we might as well try to find λ such that $A \begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 1 \end{bmatrix}$.]

   Solution note: The matrix equation tells us that $x + 1 = \lambda x$ and $x = \lambda$. Plugging the second equation into the first yields $\lambda^2 = \lambda + 1$ Solving for λ yields

   $$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

   So let $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

8. Now find $\vec{v}_1$ and $\vec{v}_2$ which have the desired property as in (6). There are infinitely many solutions. Let's normalize ours so the second component of each is 1. *(Hint: it's probably better to leave everything in terms of λ_1 and λ_2 rather than writing out what those are).*

> *Solution note:* From the previous computation, we see $\vec{v}_1 = \begin{bmatrix}\lambda_1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}\lambda_2\\1\end{bmatrix}$ works.

9. Let $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$. What is the $\mathfrak{B}$-matrix B of A? What is the change-of-basis matrix S from $\mathfrak{B}$ to the standard basis? And what is the relationship between B, S, and A?

> *Solution note:* Well, $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$, so $[A\vec{v}_1]_{\mathfrak{B}} = \begin{bmatrix}\lambda_1\\0\end{bmatrix}$, with a similar result for the second column. We get $B = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$.
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> As far as S, if we let $\mathfrak{E}$ be the standard basis, we have
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> $$S = \begin{bmatrix}[\vec{v}_1]_{\mathfrak{E}} & [\vec{v}_2]_{\mathfrak{E}}\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} = \begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}.$$
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> The relationship between B, S, and A is given by theorem 3.4.4: namely, $S^{-1}AS = B$.

10. Compute a closed formula for F_n in terms of λ_1 and λ_2.

> *Solution note:* Well, $B^n = \begin{bmatrix}\lambda_1^n & 0\\0 & \lambda_2^n\end{bmatrix}$. And inverting S gives
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> $$S^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}.$$
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> So
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> $$\begin{aligned}A^n = SB^nS^{-1} &= \frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1^n & 0\\0 & \lambda_2^n\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}\\ &= \frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_1^{n+1} & \lambda_2^{n+1}\\\lambda_1^n & \lambda_2^n\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}.\end{aligned}$$
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> Now F_n is the top left entry of A^{n-1}, i.e. the top left entry of
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> $$\frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_1^n & \lambda_2^n\\\lambda_1^{n-1} & \lambda_2^{n-1}\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix},$$
>
> which is
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> $$F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n).$$

11. Compute F_{51}.

> *Solution note:* It's $\frac{1}{\sqrt{5}}(\lambda_1^{50} - \lambda_2^{50}) = 12586269025$. (The second term is negligible - it's roughly 1.6×10^{-11}.)

12. What is the limit of F_n/F_{n-1} as $n \to \infty$? For fun, look up "Golden Ratio" on wikipedia.