Math 217: Bases and Dimension
Professor Karen Smith

A. Linear Independence.

1. Are the vectors $\vec{v}_1 = \begin{bmatrix}1 & 2 & -1\end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T$ linearly independent? Why or why not?

2. Now, consider the vectors $\vec{v}_1 = \begin{bmatrix}1 & 2 & -1\end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T$, $\vec{v}_3 = \begin{bmatrix}-1 & 0 & -1\end{bmatrix}^T$, and $\vec{v}_4 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$. Which is the first redundant vector? Write it as a linear combination of the preceding vectors.

3. Write a nontrivial linear relation among $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$.

B. Crucial Definitions: Let $\vec{v}_1, \vec{v}_2, \ldots v_d$ be vectors in a vector space V.
State precisely what it means that $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_d\}$ **spans** V.

State the precise definition of **linear independence** of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_d$.

State precisely what it means that $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_d\}$ is a **basis** for V.

Define the **dimension** of a vector space.

C. Prove: The set $\{\vec{v}_1, \vec{v}_2\}$ is LINEARLY INDEPENDENT if and only if $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$ and $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$.

D. Bases. With your table, make sure you know why the given set W is a **vector space.** Then determine whether the given set S **spans** the given vector space W. If not, what is the **subspace spanned by** S? Are the elements of S **linearly independent** ? If not, find a **relation.** Is the set $\mathcal{B}$ a **basis** for the given subspace W? What is the **dimension** of W?

1. $W = \mathbb{R}^4$, $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 + \vec{e}_1\}$

2. $W = \mathbb{R}^4$, $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_2 + \vec{e}_1\}$

3. W is the image of the linear map $\mathbb{R}^4 \to \mathbb{R}^3$ sending $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x - y \\ 2y + z \\ x + y + z \end{bmatrix}$, $S = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}$.

4. $W = \{\vec{x} \in \mathbb{R}^3 \,|\, \vec{x} \cdot \vec{u} = 0\}$ where $\vec{u} = [1 \ \ -1 \ \ 0]^T$, $S = \{[0\ 0\ 1]^T, \ \ [1\ 1\ \ 1]^T\}$.

5. W is the kernel of a surjective linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$, and S is the set consisting of just one element of $\vec{v}$ such that $T\vec{v} = 0$. [Hint: Rank-Nullity!]

6. S is the set of columns of $\begin{bmatrix} 1 & 2 & 3 & 5 & 0 \\ 5 & 4 & 9 & 13 & 6 \\ 7 & 8 & 15 & 23 & 6 \end{bmatrix}$ matrix and W is the space they span.

7. W is the image of an invertible linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ and S is the set of standard unit vectors in $\mathbb{R}^n$.

8. W is the solution space of the equations $x+y+z = 0$, $x-y+2z = 0$, $S = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$.

9. S is the set of columns of the matrix of an injective linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$, and W is the image of T.