Math 295 Daily Update

Here I will list the topics discussed in class each day.

The final exam is WEDNESDAY DECEMBER 15 at 4pm in our usual room.

- Monday 12/13: We enjoyed some Mrs Field's provided by DJ! Thanks DJ! We then continued our discussion of the construction of $\mathbb{R}$. Last time, we had defined $\mathbb{R}$ as the set of equivalence classes of Cauchy sequences of rational numbers. Today we discussed, by analogy, how to think of rational numbers as equivalence classes of pairs of integers. We then began proving that $\mathbb{R}$, as we defined it, is a *complete ordered field.* If you understand the definition well, this is mostly straightforward—addition and multiplication are defined by term-wise addition and multiplication on any representing Cauchy sequences. One subtlety is to make sure that this definition of addition and multiplication does not depend on the choice of Cauchy sequence representing a real number! The positive numbers are then those that can be represented by Cauchy sequences of positive bounded away from zero, or equivalently, those not equivalent to zero and represented by sequences of positive numbers. It remains to prove that $\mathbb{R}$ is complete: that every non-empty bounded above subset of $\mathbb{R}$ has a least upper bound. Here is the proof:

  Let $\beta \in \mathbb{R}$ be an upper bound for A. Say β is represented by a Cauchy sequence of rational numbers $\{b_n\}$. Since Cauchy sequences are bounded above, there is some $b \in \mathbb{Q}$ bounding all the b_n. Without loss of generally, then, we can assume $\beta = b$ is rational (thinking of a rational number as represented by a constant cauchy sequence). Similarly, since A is non-empty, it contains some α, which is in turn bounded below by some rational a. Set $l_0 = a$ and $u_0 = b$, and let m_0 be halfway in between: $m_0 = \frac{l_0+u_0}{2}$. Note that m_0 is rational as well! (DRAW A PICTURE!). There are two possibilities. Either m_0 is an upper bound for A or not. If m_0 is an upper bound, let $u_1 = m_0$ and $l_1 = l_0$. If m_0 is NOT an upper bound, let $l_1 = m_0$ and $u_1 = u_0$. The midway point of these $m_1 = \frac{l_1+u_1}{2}$ is again rational, and either an upper bound for A or not. Inductively we continue, constructing a sequence $m_n = \frac{l_n+u_n}{2}$ where $u_n = u_{n-1},\ l_n = m_{n-1}$ if m_{n-1} is not an upper bound for A and $u_n = m_{n-1},\ l_n = l_{n-1}$ if m_{n-1} is an upper bound for A. In this way we construct Cauchy sequences
  $$\{l_n\} \leq \{m_n\} \leq \{u_n\}.$$
  Since $u_n - l_n = \frac{u_0-l_0}{2^{n-1}}$ we immediately verify that these Cauchy sequences are all equivalent Cauchy sequences of rational numbers representing therefore the same real number, call it μ. Because all the u_n are upper bounds for A, it is easy to see that μ is an upper bound for A. But also because no l_n is an upper bound for A, it is easy to see that no smaller number can bound A. Thus A has a least upper bound. This shows $\mathbb{R}$ is complete. So $\mathbb{R}$ exists!

- Friday 12/10: We proved that every Cauchy sequence of real numbers converges *in* $\mathbb{R}$. The point is the *completeness axiom of* the real numbers—this result is false in $\mathbb{Q}$. We then **constructed** the real numbers (which so far, has been an undefined concept!) as the set of all Cauchy sequences of rational numbers—the idea being that a real number can be represented as a Cauchy sequence of rational numbers. By declaring a real number to *be* such a Cauchy sequence, we can make the idea of a real number precise—it is no longer undefined. You are familiar with this, since you have been identifying real numbers with "decimal approximations" since middle school. By taking more and more decimals places, we get one way of finding a Cauchy sequence of rational numbers representing a real number. However, just as $.9999\cdots = 1.0000\ldots$, we need to identify any two Cauchy sequences whose difference converges to zero—these two sequences ought to converge to the same limit. We did this by introducing an *equivalence relation* on the set of all Cauchy sequences of $\mathbb{Q}$. Next time we will show that this set $\mathbb{R}$ we have constructed is a complete ordered field.
- Wednesday 12/8: Quiz. We continued discussing Cauchy sequences. We showed that it is not enough, in the definition of Cauchy, to require that the difference of successive terms is eventually less than ϵ. The example was the harmonic series. We showed that convergent sequences are Cauchy and Cauchy sequences are bounded in any ordered field. We proved, using the completeness axiom, that bounded monotone sequences of real numbers always converge.
- Tuesday 12/7: We defined a sequence of elements in any set X as a function $\mathbb{N} \to X$. The image of n is denoted a_n, so the sequence is often denoted $\{a_n\}_{\{n\in\mathbb{N}\}}$. The main case for us is when X is an ordered field, in which case we can define what it means for a sequence to have a limit, and also a cauchy sequence, which is the formal mathematical way to say that the elements of the sequence are "clustering together" as n goes to infinity. **Assignment: Read Chapter 22. Prepare for quiz on definition of limit of sequence and definition of Cauchy sequence.**
- Monday 12/6: We studied the properties of $exp(x)$ directly from the definition as the function which is the inverse of ln. In particular, we saw that $exp(x) = e^x$ where $e = exp(1)$, for all rational numbers x. We can then interpreting e^x for non-rational x as $exp(x)$, which, since exp is continuous and increasing, must be the supremum of the e^q as we range over all rational $q < x$. Alternatively, approximating x by sequence q_n of rational numbers approaching x, the value of e^x is necessarily the limit of e^{q_n}. We will make this idea precise tomorrow when we talk about limits of sequences.
- Friday 12/3: We defined the natural logarithm as $ln\, x = \int_1^x \frac{1}{t}dt$ for any positive real x. Straight from this definition we showed that ln is an increasing function satisfying $ln\, 1 = 0$, differentiable, whose derivative at x is $\frac{1}{x}$. We also proved other basic properties such that $ln(xy) = ln(x) + ln(y)$, and that $ln(x)$ is unbounded above and below, and that its range is all of $\mathbb{R}$. This implies that ln is a bijective

function from $(0, \infty)$ to $\mathbb{R}$, and therefore has an inverse function, which is called the exponential function, denoted exp. So $exp : \mathbb{R} \to (0, \infty)$ is also differentiable, and using the chain rule on $ln(exp(x)) = x$, one easily derives that $exp'(x) = exp(x)$. **Assignment: Read Chapter 18 on exponential functions.**

- Wednesday 12/1: We gave a cute proof (using the mean value theorem) that if f is integrable on $[a, b]$ and it is the derivative of some function g, then $\int_a^b f = g(b) - g(a)$. Note that we do not require f to be continuous! However, in the special case where f is continuous, this can also be deduced as an easy corollary of the fundamental theorem of calculus proved last time. We also defined the cantor set and cantor function. John Holler typed up this lecture; please read it! (Link from the 295 page). **Assignment: Read Chapter 18 on exponential functions.**
- Tuesday 11/30: We proved the fundamental theorem of calculus. Note: be sure to read Chapter 14, which has some important corollaries and variants of the FTC, as well as a different proof (using the Mean Value Theorem) of the corollary stated at the end of class: if f is integrable on $[a, b]$ and there is function g such that $g' = f$, then $\int_a^b f = g(b) - g(a)$. **Assignment: Reread Chapter 14**
- Monday 11/29: We talked about the fundamental theorem of calculus: how integration can be throught of as an operator on the space of integrable functions, and actually “improves” functions— the integral of an integrable function is continuous, the integral of a continuous function is differentiable. We thought some about different classes of functions which all have an “addition” and ”scalar multiplication” defined on them, and how differentiation and integration can be thought of as linear operators on these spaces. **Assignment: Read Chapter 14 and the last two pages of Chapter 13.**
- Wednesday 11/24: We discuss the topology and geometry of the real projective plane. **Assignment: Be thankful. Happy Thanksgiving!**
- Tuesday 11/23: We proved that the inverse f^{-1} of a bijective function $f : A \to B$ (where A and B are subsets of $\mathbb{R}$) is continuous at $b = f(a)$ if f is continuous at a. The proof uses the idea of connectivity from our topological discussions and is framed differently than in the book. We also discussed that f^{-1} is differentiable at $b = f(a)$ if f is differentiable at a and $f'(a)$ is not zero. In this case, we used the chain rule to see that $(f^{-1})'(b) = \frac{1}{f'(a)}$. Using this, we derived the rule for differentiating $f(x) = x^{\frac{1}{n}}$ and also $f(x) = x^{\frac{m}{n}}$. **Assignment: READ CHAPTER 12.**
- Monday 11/22: EXAM II.
- Friday 11/19: We discussed many theorems from Chapter 11, including the mean value theorem and the fact that if a function f defined on an open interval has a maximum at m and is differentiable at m, then $f'(m) = 0$. John Holler wrote up nice notes for the Chain rule discussion: PLEASE READ THEM! They are on the

course page. There are many good theorems in this chapter you are responsible for, please read it! **Exam in class MONDAY NOV 22.**

- Wednesday 11/17: Chain Rule! We tried to really understand *why* it is true, and proved it precisely by approximating a differentiable function by a linear function in a neighborhood of a. The point is that when you compose linear functions, the slopes multiply. Since a differentiable function f can be approximated by a linear function at each point, we can deduce the chain rule from this simple fact about linear functions. **Read Chapter 11. Work on the Exam Problems; you will write four of them in class MONDAY NOV 22.**
- Tuesday 11/16: We proved (slightly differently than the book) that if a function is differentiable at a, then it is continuous at a. We discussed some of the proofs of the basic "differentiation rules" and proved the product rule. **By now you should have read Chapters 9 and 10. Start reading Chapter 11. Work on the Exam Problems; you will write four of them in class MONDAY NOV 22.**
- Monday 11/15: We defined what it means for a function (from $\mathbb{R}$ to $\mathbb{R}$) to be differentiable at a point of its domain, and the derivative function of a differentiable function. We stated the theorem if f is differentiable at a, then it is continuous at a. We are following Chapter 9 closely. **Read Chapter 10, on computing derivatives. Work on the Exam Problems; you will write four of them in class MONDAY NOV 22.**
- Friday 11/10: We discussed Riemann integration. We defined Riemann integrability and proved that a Riemann integrable function is bounded. We discussed the fact that Riemann and Darboux integrability are equivalent. **Read Chapter 9, on the derivative. Also, begin working on the Exam Problems; the exam will be in class MONDAY NOV 22.**
- Wednesday 11/10: **Reread Chapter 13!** We continued discussing integrability, following book closely. We reproved the theorem that a continuous function on a compact set is integrable. The proof used uniform continuity.
- Tuesday 11/9: **Reread Chapter 13!** We continued discussing integrability, following book closely. We also began class with a slightly off topic but cool fact in response to the question monday (whose?): how big is the set of rationals are there relative to the reals? We showed that the rationals can be covered by a union of open intervals whose size sum to an *arbitrarily small* number!
- Moday 11/8: **Read Chapter 13!** We will be following the book closely, and lectures will assume you have read ! Today: We defined partitions, refinements of partitions, upper and lower sums with respect to a partition, and what it means for a function to be integrable—basically page 250 to midway through 256.
- Friday 11/5: We took a T/F quiz on basic topology. We then discussed uniform continuity, and proved that continuous functions on compact sets are uniformly continuous. We then started talking about integrals. We defined partitions of an interval,

and upper and lower sums for a bounded function with respect to a partition. **Read Chapter 13, at least through page 262 by monday, through end by tuesday.**

- Wednesday 11/3: We summarized all the theorems we've proved in general topology, including as special cases the "Three hard theorems" from Chapter 7. We discussed why compactness is so important in mathematics, and gave an example: If a function $f : X \to \mathbb{R}$ is locally bounded on a compact set, then it is bounded. Understanding the proof of this is crucial to see how compactness gets used in mathematics. **Quiz Friday on topology definitions.**
- Wednesday 11/3: We summarized all the theorems we've proved in general topology, including as special cases the "Three hard theorems" from Chapter 7. We discussed why compactness is so important in mathematics, and gave an example: If a function $f : X \to \mathbb{R}$ is locally bounded on a compact set, then it is bounded. Understanding the proof of this is crucial to see how compactness gets used in mathematics. **Quiz Friday on topology definitions.**
- Tuesday 11/2: We proved that a closed bounded set in $\mathbb{R}$ is compact. **Assignment: READ THE NOTES! Also, read the appendix to Chapter 8 on uniform continuity.**
- Monday 11/1: We discussed compactness, one of the most important and interesting ideas in mathematics. We gave many examples of compact and non compact sets. We showed that a compact subset of $\mathbb{R}$ is always closed and bounded—-The converse is also true, though we have not yet shown it. We prove a theorem that the image of a compact set under a continuous map is always compact, and deduced as a corollary the extreme value theorem from Chapter 7: If $f : [a, b] \to \mathbb{R}$ is a continuous function, then there exists x_m and x_M in $[a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$. **READ THE NOTES!**
- Friday 10/29: We reproved the "generalized IVT:" The image of a connected set under a continuous map is connected. We then defined open covers and subcovers, for any subset of a topological space, and gave many examples. We defined compact sets as those with the property that every open cover has a finite subcover. **READ THE NOTES!**
- Wednesday 10/27: We proved the intermediate value theorem as a corollary of a very general theorem about continuous maps of topological spaces. **READ THE NOTES!**
- Tuesday 10/26: We proved that the composition of continuous maps is continuous (using the abstract point of view, it was easy!). We defined a *connected* topological space. We showed that $\mathbb{R}$ is connected in the Euclidean topology but that $\mathbb{Q}$ is not (in the subspace topology). Stayed tuned for Lecture Notes—as soon as I have and edit them from John, they will be up on the course page.
- Monday 10/25: We continued talking about abstract topological spaces, and defined the subspace topology as well as continuous maps. *Please see the class notes*

typed up by John Holler on the course web site. **Assignment: Read Course Notes, report typos. Also, download and read Handout 6 on Closed sets.**

- Friday 10/22: We discussed one important and natural class of examples of *abstract topological spaces:* those coming from a *metric* (or distance function) on a set X. That is, we defined abstract metric spaces and showed how each gives rise to a topology. The main examples are the Euclidean spaces: $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, and indeed $\mathbb{R}^N$, for any natural number N, with the usual notion of (Euclidean) distance. We discussed some other examples: the *taxicab metric* on $\mathbb{R}^2$, the *trivial topology* on any set X, the *discrete topology* on any set X, and various (not very natural) topologies on the set of 295 students. We also defined *closed sets* in a topological space and looked at which kinds of intervals in $\mathbb{R}$ are open, closed or neither. **Assignment: Read Chapter 7. Our goal is to use topology to give "easy" proofs of these three hard theorems, which Spivak proves using analysis (meaning $\delta - \varepsilon$ calculus).**
- Wednesday 10/20: We finished the proof that a function $f : \mathbb{R} \to R$ is continuous if and only if the preimage of open sets is open. We began discussing the notion of an *abstract topological space.* The example of $\mathbb{R}$ (with the usual notion of openness) and higher dimensional Euclidean space $\mathbb{R}^N$ were given. For the latter, we had to define a notion of distance in $\mathbb{R}^N$, which gave us a notion of an "ϵ-ball," and hence of open set (using the same definition of open set we have for $\mathbb{R}$.)
- Friday 10/15: Exam I. High Score 100, Low Score 21, Median Score 76.5. More details on Course Page, click on "Exam I Results."
- Wednesday 10/13: We talked more about the *image* and *pre-image* of sets under any function $f : A \to B$. We looked at more examples. We proved one direction of the theorem about continuity stated last time.
- Tuesday 10/12: We defined open sets, and looked at lots of examples. We defined the *pre-image* of a set under any function. We stated a theorem describing continuity: a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all open sets U in the range, $f^{-1}(U)$ is open in the domain.
- Monday 10/11: We carefully stated and proved the following: "if the limit of f as x approaches a is L and the limit of g as x approaches a is M, then the limit (at a) of the product of functions f and g is LM." We observed that this proves the product of continuous functions is continuous. Together with what we did last time, we concluded that polynomials are always continuous. **Assignment: Reread Chapters 5 and 6.**
- Friday 10/8: We practiced some basic logic, negating "for all there exists" statements. Students who did not get problem 2 correct on the quiz are expected to do the "logic practice sheet" on the 295 homepage. We defined continuity of functions, studied examples of discontinuous functions with holes, breaks, and wild oscillations. We stated a theorem that the sum, product, and quotient of continuous functions is continuous, and started proving it. **Assignment: Read Chapter 6.**

- Wednesday 10/6: We took a quiz. We then practiced the delta-epsilon definition of a limit in a few examples. **Assignment: Reread Chapter 5! Also: Negate this sentence: For all boys, there exists a girl who has written him a love note. Now use the symbol B for the set of boys and G for the set of girls, and write this sentence and its negation using mathematical symbols.**
- Tuesday 10/5: We practiced the definition of a limit, from analytic and topological point of view. We showed some simple linear and quadratic functions have limits from the definition. We proved the limit of any function at any point is unique, if it exists. **Assignment: Memorize the precise definition of a limit.** *There was some confusion in lecture about the problem 3a from chapter 5. Some students argued that the professor should have taken $\delta = \frac{\epsilon}{8}$ instead of $\delta = \frac{\epsilon}{10}$. The professor was correct, though apparently she admits, did not do a good enough job of explaining. For sure, taking inputs from a smaller δ-neighborhood of 2 only makes it easier to land in the desired ϵ-neighborhood of 12. So if $\delta = \frac{\epsilon}{8}$ works, so does $\frac{\epsilon}{10}$. What some students were concerned with (I think) in bringing up the $\frac{\epsilon}{8}$, was trying to find the largest possible δ that might work. Of course, if those students can show that $\frac{\epsilon}{8}$ also works, that is, whenever x is in a punctured $\frac{\epsilon}{8}$-neighborhood of 2, then the outputs values of f are all in the ϵ-neighborhood of 12, their argument is also correct. The point is to find any δ that works—do not get hung up on "finding the best δ." Shrink δ as needed, whenever it simplifies your life. Trying to find the best δ may lead to difficult arithmetic, or be almost impossible in practice. Good analysts use tricks to simplify estimates; you should too.*
- Monday 10/4: We discussed the Contiuum hypothesis, and the fact that it has been proved independent of the "standard axioms" of mathematics. We then started **Calculus** by giving the precise definition (delta-epsilon definition) of a limit. **Assignment: Finishing reading/reread Chapter 5. It would be a good idea to try your skill on the delta-epsilon definition as it will definitely by on next week's exam. For example, can you do problem 3a from Chap 5? Try it, I will go over it in class.**
- Friday 10/1: We sketched the proof that the rational numbers are countable; details are on Assignment 4. We discussed Cantor's diagonal argument that there exist uncountable sets; specifically, we showed the set of sequences of zeros and ones is uncountable. **Assignment: Monday we start Calculus! Start reading Chapter 5 (on limits), at least through page 100 by Monday and all by Tuesday.)**
- Wednesday 9/29: We proved: *The following are equivalent for a function $A \xrightarrow{f} B$: (1) f is bijective; (2) f^{-1} is a function; (3) there exists a function $B \xrightarrow{g} A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. Furthermore, in this case, the function g is unique and equal to f^{-1}.* We then defined countability, and what it means for one set to have the same or larger cardinality than another. **Assignment: Read Chapter 12 through first paragraph on p 234. Why does Theorem 1 on page 231 appear to contradict the theorem proved in class?**

- Tuesday 9/28: Proof practice. We proved: *If $a, b \in \mathbb{R}$ with $b - a > 1$, then there exists an $n \in \mathbb{Z}$ such that $a < n < b$.* More on functions and graphs, an example of a bijective map between a set S and a proper subset.
- Monday 9/27: More on functions, the inverse $f^{-1} \subset B \times A$. Fact that f^{-1} is a function if and only if f is bijective. more examples. **Assignment: Read Chapter 4.**
- Friday 9/24: Proof that $\mathbb{R}$ is archimedian. Formal definition of a function as a (certain kind of) subset of $A \times B$. Vocabulary: injective, surjective, bijective functions, domain (or source) and range (or target).
- Wednesday 9/22: Example of a proof involving supremums. Archimedian fields. An example of a non-archimedian ordered field F, in which $\mathbb{N}_F$ is a *bounded* subset!
- Tuesday 9/21: Well-ordering. Division algorithm. **Assignment: Read Chapter 3.**
- Monday 9/20: More on Induction: inductive sets, strong induction. proof that every integer greater than one is a product of finitely many primes.
- Friday 9/17: Mathematical Induction. **Assignment: Read Chapter 2.**
- Wednesday 9/15: The completeness property of the real numbers. Complete ordered fields. **Assignment: Reread everything from chaps 1 and 28 (and part of 8). We are moving on.** *Optional Assignment: Read chapter 29 (the construction of the real numbers) and chapter 30 (the proof that there is only one complete ordered field, up to isomorphism.)*
- Tuesday 9/14: Absolute value in any ordered field. Begin discussion of greatest lower and least upper bounds. **Assignment: Download and read Handout on absolute value. Read Chapter 8, to middle of page 13 (stop at theorem 7.1).**
- Monday 9/13: Definition of greater than and less than in any ordered field, basic properties of ordered fields. **Assignment: Reread everything so far. How many fields of two elements can there be?**
- Friday 9/10: More discussion of fields, proofs of some basic properties in fields, the even-odd field, definition of ordered structure. **Assignment: Download Handout on how to write proofs. Discuss with classmates the even-odd field.**
- Wednesday 9/8: Arithmetic properties of real numbers. Binary operations on sets. associative, commutative properties. identity elements, inverse elements. Definition of a field. examples. **Assignment: Read Chapter 28. Read Chapter 1 again if needed. Find classmates to work with. Start thinking of homework.**
- Tuesday 9/7: Course overview, intermediate value theorem, natural, rational and real numbers. What is a real number? Proof that $\sqrt{2}$ is not rational. **Assignment: Read Chapter 1, find professor's website and download handout 1.**