When doing the problems, you may use your notes, your book, and all notes and handouts on the professor’s webpage, but nothing else. This means: do not use the Internet, your classmates, your roommates, your parents, math books (other than your text), your acquaintances, etc. The midterm will include four of these problems; there will be some choice of which four.

Unless otherwise indicated, you may use without proof any result proved in class or in homework. If you do this, you must state clearly the result that you are using. The exam itself will be closed book.

During the exam, you will be asked to sign the following pledge: I pledge my honor that I have not used or sought mathematical assistance from any source apart from the course text and my class notes in connection with my work on this examination.

(1) For each item below, find an example, or prove that none exists:
   (a) An open cover of \((-1, 1)\) which has no finite subcover.
   (b) An open cover of \([0, 10]\) which has no finite subcover.
   (c) A function on \((0, 1)\) which is continuous but not uniformly continuous.
   (d) A function on \([-1, 1]\) which is continuous but not uniformly continuous.
   (e) A non-constant uniformly continuous function on the real line.
   (f) A non-empty proper connected subset of \(\mathbb{R}\) (in the Euclidean topology) which is not an interval.
   (g) A compact subset of \(\mathbb{R}\) which is not bounded.
   (h) An unbounded continuous function on \([0, 100]\).
   (i) A continuous function on \([0, 3]\) which is not integrable.
   (j) A non-continuous function on \([0, 4]\) which is integrable.

(2) Let \(f : [a, b] \to \mathbb{R}\) be a bounded function. Show that if \(f\) is integrable, then \(|f|\) is integrable. Is the converse true? (Hint: Compare \(U(|f|, P) - L(|f|, P)\) and \(U(f, P) - L(f, P)\).)

(3) Let \(f : [a, b] \to \mathbb{R}\) be a bounded integrable function. Show that

\[
\left| \int f \right| \leq \int |f|.
\]

(4) Find an example of a connected topological space with exactly five points and exactly six open sets. Find an example of a disconnected topological space with exactly five points and exactly six open sets. (Or, in either case, prove none exists.)
(5) **Topological Spaces via closed sets.**

(a) Prove de Morgan’s Laws: If \( \{A_\lambda\}_{\lambda \in \Lambda} \) is a collection of subsets of a set \( X \), then

(i) \( \bigcup_{\lambda \in \Lambda} (X \setminus A_\lambda) = X \setminus \bigcap_{\lambda \in \Lambda} A_\lambda \).

(ii) \( \bigcap_{\lambda \in \Lambda} (X \setminus A_\lambda) = X \setminus \bigcup_{\lambda \in \Lambda} A_\lambda \).

(b) Let \( X \) be a set, and let \( \mathcal{C} \) be a collection of subsets of \( X \) satisfying the following conditions:

(i) \( X \) and \( \emptyset \) are in \( \mathcal{C} \)

(ii) If \( C_1, \ldots, C_t \) is a finite collection of sets in \( \mathcal{C} \), then the union \( \bigcup_{i=1}^{t} C_i \) is in \( \mathcal{C} \).

(iii) If \( \{C_\lambda\}_{\lambda \in \Lambda} \) is an arbitrary collection of sets in \( \mathcal{C} \), then the intersection \( \bigcap_{\lambda \in \Lambda} C_\lambda \) is in \( \mathcal{C} \).

Let \( \mathcal{O} \) be the collection of subsets of \( X \) whose complement is in the collection \( \mathcal{C} \). Prove that \( \mathcal{O} \) forms a topology on \( X \).

(c) Show that the co-finite topology on any set is a topology. (The cofinite topology on a set \( X \) has open sets that are the complements of finite sets, plus of course \( X \) and \( \emptyset \).)

(6) Prove or disprove: if \( f \) and \( g \) are uniformly continuous and \( g \) is not zero anywhere in the domain, then the quotient \( f/g \) is uniformly continuous.

**Book Problems** Chapter 2: 6(i), Chapter 13: 1, 5(i), 13, 20, 34