The final will be two hours long. It will have six questions. Four of these will be taken
directly from the first eight problems below. The remaining two will not be taken from the
problems but will be similar to the sample problems provided (a “short answer” type problem
and an essay).

When doing test problems, you may use your notes, your book, and all notes and hand-
outs on the professor’s webpage, but nothing else. This means: do not use the Internet,
your classmates, your roommates, your parents, math books (other than your text), your
acquaintances, etc. Seeking help from any unauthorized source is cheating.

Unless otherwise indicated, you may use without proof any result proved in class or in
homework. Please carefully state the theorems you use, and show how to use them to
get the results you need. Most of the problems below follow fairly quickly from the big
theorems we proved: you are not expected to reprove them! The exam itself will be closed
book.

During the exam, you will be asked to sign the following pledge: I pledge my honor that I
have not used or sought mathematical assistance from any source apart from the course text
and my class notes in connection with my work on this examination.

1. Show that every closed subset of a compact topological space is compact.
2. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that there exists an element $c \in (a, b)$ such
   that $\int_a^b f(t) dt = (b - a) f(c)$.
3. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if and only if for each sequence $\{x_n\}$
in $\mathbb{R}$ which converges to $a$, the sequence $\{f(x_n)\}$ converges to $f(a)$.
4. Let $h : \mathbb{R} \to \mathbb{R}$ be continuously differentiable at a point $c$ in the domain (this means
   that $h'$ exists and is continuous at $c$).
   (a) Prove that if $h'(c)$ is not zero, then $h$ has an inverse function in a neighborhood
       of $c$.
   (b) Prove that if $h'(c)$ is not zero, then $h^{-1}$ is differentiable at $h(c)$, and express its
       derivative at $h(c)$ in terms of $h'(c)$.
   (c) Prove that the derivative of the exponential function $exp$ is defined at all $x$ in $\mathbb{R}$
       and equal to $exp(x)$.
5. Let $f_1$ and $f_2$ be continuous functions on the real line. Define new functions $h_1$ and
   $h_2$ on $\mathbb{R}$ as follows:
   \[ h_i(x) = \int_0^x f_i(t) dt, \text{ for } i = 1, 2. \]
   (a) Show that the product function $h = h_1 h_2$ is differentiable at every point of its
domain.
(b)Give a formula for \( h'(c) \) in terms of \( f_1(c), f_2(c), h_1(c) \) and \( h_2(c) \).

(6) Let \( \{a_n\} \) and \( \{b_n\} \) be Cauchy sequences of elements in an ordered field. Prove that

(a) The sequence \( \{a_n\} \) can have at most one limit.
(b) If a subsequence of \( \{a_n\} \) converges to \( \alpha \), then also \( \{a_n\} \) converges to \( \alpha \).
(c) The sum sequence \( \{s_n\} \) where \( s_n = a_n + b_n \) is Cauchy.
(d) (Optional: not on exam) The product sequence \( \{p_n\} \) where \( s_n = a_n b_n \) is Cauchy.
(e) (Optional: not on exam) If the limit of \( \{a_n\} \) is not zero, then \( \{a_n\} \) has a subsequence whose sequence of reciprocals \( \{r_n\} \) where \( r_n = \frac{1}{a_n} \) is Cauchy.  

(7) Let \( f: [0, \infty) \rightarrow (0, \infty) \) be a function whose restriction to any closed interval in its domain is integrable. For \( x \geq 0 \), define \( F(x) = \int_0^x f \).

(a) Show that \( F \) is an injective function.
(b) Show that the image of \( F \) is either \([0, a)\) for some \( a > 0 \) or \([0, \infty)\).
(c) Suppose \( a > 0 \). Find a function \( f \) so that the image of \( F \) is \([0, a)\).
(d) Find a function \( f \) so that the image of \( F \) is \([0, \infty)\).

(8) (a) Find (with rigorous proof) a necessary and sufficient condition on a rational number \( r \) such that

\[
\lim_{\epsilon \to 0^+} \int_{\epsilon}^1 x^r
\]

exists (meaning, approaches some real number).
(b) Using your answer to (a), prove necessary and sufficient conditions on a rational number \( r \) such that the series

\[
\sum_{i=1}^{\infty} \frac{1}{i^r}
\]

converges. (Recall that by definition, this means that the sequence of partial sums \( \{p_n\} \) where \( p_n = \sum_{i=1}^{n} \frac{1}{i^r} \) converges.)

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1Hint: these proofs are very similar to the analogous statements for limits of functions. Fix \( \epsilon > 0 \). To find the \( n \) you need is similar to finding the \( \delta \) you need when you show, for example, as we did in class, that a product of continuous functions is continuous.
The following are practice problems for the “short answer” part of the test.

For each item below, find an example, or prove that none exists:

1. A Cauchy sequence of rational numbers which does not converge.
2. A Cauchy sequence of real numbers which does not converge.
3. A compact set of real numbers which is not an interval.
4. A non-trivial, non-discrete topology on the real line which is not the Euclidean topology.
5. A continuous function \( f \) on \([0, 1]\) which is integrable, but for which there is no partition \( P \) on \([0, 1]\) such that \( L(f, P) = U(f, P) \).
6. A surjective continuous map \( f : [0, 6] \to \mathbb{R} \).
7. A set of real numbers which has a supremum but not a maximum.
8. A subset of \( \mathbb{Q} \) which witnesses the fact that \( \mathbb{Q} \) is not complete.
9. A connected subset of \( \mathbb{R} \) which is not bounded.
10. A function \( f \) on \([0, 17]\) whose derivative is continuous function \( \sin(ln(ln(|\sin(x\cos x)|)))) \).
11. An function \( g : (0, \infty) \to \mathbb{R} \) satisfying \( \lim_{x \to \infty} \int_{1}^{x} g(t)dt \) exists.

Essay. You will be asked to write an essay on some major concept of the course. There will be considerable freedom on this essay, so you will be better off being deeply prepared on a few items, rather than superficially prepared on all of them.

What does deeply prepared mean? You can precisely state the relevant definitions and/or theorems, as well as important examples (and/or non-examples) that illustrate them. You know why the concept is important, whether to calculus or beyond, and how it connected with other issues in the course. You can explain it in clear mathematically precise language in as simple as possible terms. You may not have to include all proofs of theorems you state, but in some cases, sketches of proofs, especially if they help illustrate your concept and/or are especially enlightening may be appropriate.

Some possible essay titles might be:

2. The fundamental theorem of Calculus.
3. Compactness.
4. What is \( \mathbb{R} \)?
5. Integration
6. Continuity

You might prepare by thinking through some of the major ideas of the course, making sure you can precisely state the associated definitions and theorems, and have a body of interesting examples to illustrate the ideas.