When doing the problems, you may use your notes and your book, but nothing else. This means: when working on these problems, do not use the Internet, your classmates, your roommate(s), your parents, math books (other than your text), your acquaintances, etc. The midterm will consist of four of these problems; there will be some choice of which four.

Unless otherwise indicated, you may use without proof any result proved in class or in homework. If you do this, you must state clearly the result that you are using. You may also use without proof ordinary algebraic manipulations of equations and inequalities. The exam itself will be closed book.

During the exam, you will be asked to sign the following pledge: I pledge my honor that I have not used or sought mathematical assistance from any source apart from the course text and my class notes in connection with my work on this examination.

(1) Each of the following statements is either true or false. If the statement is true, indicate this. If the statement is false, indicate that the statement is false and provide a (justified) counterexample showing why the statement is false.
   (a) Suppose $S$ and $T$ are nonempty bounded subsets of $\mathbb{R}$. If $S \subset T$, then $\inf S \leq \inf T \leq \sup S \leq \sup T$.
   (b) The composition of injective functions is injective.
   (c) If $S$ is a proper subset of $A$, then there does not exist a bijection $f : S \to A$.
   (d) For $q \in \mathbb{Q}$, the set $\{x \in \mathbb{Q} : x < q\}$ contains a maximal element.
   (e) Every bounded above nonempty subset of $\mathbb{R}$ has a least upper bound.

(2) Definition: A subset $U$ of the real numbers is open if for all $x \in U$, there exists $\epsilon > 0$ such that the $\epsilon$-ball $B_\epsilon(x) \subset U$. That is, $U$ is open if for all $x \in U$, there is an $\epsilon$-neighborhood of $x$ contained in $\mathbb{R}$.
   (a) The open interval $(a, b)$.
   (b) The closed interval $[a, b]$.
   (c) The set of rational numbers.

(3) Show that for any function $f : \mathbb{R} \to \mathbb{R}$, and any $a \in \mathbb{R}$, we have $\lim_{x \to a} f(x) = \lim_{h \to 0} f(x + h)$.

(4) Suppose $a, c \in \mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a$. Show the following.
   (a) $cf$ is continuous at $a$.
   (b) If $f(a) < 0$, then $f$ is negative on some neighborhood of $a$. (This means $f$ is negative at all points of some neighborhood of $a$.)

(5) Let $A$ be any set and suppose that a function $f : A \to A$ satisfies $f \circ f \circ f$ is the identity function on $A$. Prove or disprove that $f$ is a bijection.

(6) Suppose $x \in \mathbb{R}$. Prove: We have $x \geq 1$ if and only if for all $\varepsilon > 0$ we have $x > 1 - \varepsilon$.

(7) Give an $\varepsilon - \delta$ proof that $f(x) = 17x^2 - 19x + 23$ is continuous at 1.

(8) Suppose $a \in \mathbb{R}$ and $0 < a < 1$. Consider the set $S = \{a^n : n \in \mathbb{N}\}$. Show that $S$ is bounded above and below. If $S$ has an infimum, what is it? (Here, and everywhere on this exam, be sure to prove all your statements.)

(9) Let $f(x) = \sin \left( \frac{1}{x} \right)$. Prove or disprove: the limit of $f(x)$ as $x$ approaches 0 exists.

(10) Suppose $S$ is a nonempty, bounded above subset of $\mathbb{R}$. 
(a) Suppose $c \in \mathbb{R}$ is positive. Define $cS := \{cx \mid x \in S\}$.
Show that $\sup(cS) = c \sup(S)$.

(b) Define $-S := \{x \mid -x \in S\}$.
Show that $\inf(-S) = -\sup(S)$.

(11) (Left and right hand limits) Suppose $a, \ell \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a function. We say that $f(x)$ approaches $\ell$ as $x$ approaches $a$ from the right (or above) provided that for all $\varepsilon > 0$, there is a $\delta > 0$ so that if $0 < x - a < \delta$, then $|f(x) - \ell| < \varepsilon$. When this happens, we write $\lim_{x \to a^+} f(x) = \ell$; if this fails to hold, then we say that $\lim_{x \to a^+} f(x)$ does not exist.
(a) Find a function for which the $\lim_{x \to 0} f(x)$ does not exist, but $\lim_{x \to 0^+} f(x) = 5$.
(b) Write down the definition of what we mean when we say “$f(x)$ approaches $\ell$ as $x$ approaches $a$ from the left (or below)”. In this case, we will use the notation $\lim_{x \to a^-} f(x) = \ell$.
(c) Prove that if $\lim_{x \to a^+} f(x) = \ell$ and $\lim_{x \to a^-} f(x) = \ell$, then $\lim_{x \to a} f(x) = \ell$. Is the converse true?

(12) (Horizontal Asymptotes) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that $f$ approaches $L$ as $x$ approaches $\infty$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$. In this case we say that $\lim_{x \to \infty} f(x) = L$. Show that $f(x) = \frac{2x-1}{x+3}$ approaches some $L$ as $x$ goes to infinity.