Math 295. Handout on absolute values

The following result was stated in class without proof. Here we give the "mathematician's conceptual proof", rather than a brute-force case-by-case proof. The context is of course so elementary that brute-force is perfectly fine, so this should merely be viewed as an illustration for how a more conceptual point of view can sometimes lead to more elegant arguments that use "pure thought" rather than merely explicit calculation. The proof of (ii) below is essentially the one in the book, except the book neglects to point out the use of (i) (which the book forgets to prove).

Theorem. *Let F be an ordered field, $a, b \in F$. Then the following hold:*
(i) $|ab| = |a||b|$,
(ii) $|a+b| \leq |a| + |b|$

Proof. Recall from HW 1, Exercise 2(iii) (whose proof does *not* use the concept of absolute value, thereby avoiding the risk of circular reasoning) that if $x, y \in F$ and $x, y \geq 0_F$, then $x^2 \leq y^2$ if and only if $x \leq y$, with equality in either case forcing equality in the other case. Since both sides of (i), (ii) are non-negative, it therefore suffices to check each assertion after squaring both sides. That is, we want to prove
(i') $|ab|^2 = (|a||b|)^2$,
(ii') $|a+b|^2 \leq (|a| + |b|)^2$.

Since $|x|^2 = x^2$ for all $x \in F$, we compute $|ab|^2 = (ab)^2 = a^2b^2$ and $(|a||b|)^2 = |a|^2|b|^2 = a^2b^2$. This settles (i'). For (ii'), we compute

$$|a+b|^2 = (a+b)^2 = a^2 + 2_F ab + b^2$$

and

$$(|a| + |b|)^2 = |a|^2 + 2_F|a||b| + |b|^2 = a^2 + 2_F|ab| + b^2,$$

where the final equality uses the *already proven* (i). Thus, (ii') reduces to the claim

$$a^2 + 2_F ab + b^2 \leq a^2 + 2_F|ab| + b^2. \tag{1}$$

Since $x \leq |x|$ for all $x \in F$, we know $ab \leq |ab|$. But $2_F > 0_F$, so $2_F ab \leq 2_F|ab|$ also. From this, (**??**) follows.

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