Math 295. Handout 6 on Closed Sets

The purpose of this handout is to justify a geometric description of closed sets as those which are "stable under formation of limits." The idea is that to check if $C \subseteq \mathbb{R}$ is closed, it should be equivalent to say that for arbitrary $x \in \mathbb{R}$, $x \in C$ if and only if x can be approximately to arbitrarily high accuracy by an element of C. Think of the closed sets $[0, 1]$ and $\{0\} \cup \{1/n \,|\, n \in \mathbb{N}\}$, and the non-closed set $\{1/n \,|\, n \in \mathbb{N}\}$ for which 0 is *not* in the set but can be approximated to arbitrarily high accuracy by suitable elements $1/n$.

Put more precisely, we make a definition. Let $C \subseteq \mathbb{R}$ be a subset. We say that $x \in C$ is an *accumulation point* of C if, for all $\varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon)$ meets C. That is, for all $\varepsilon > 0$ there should exist some $c \in C$ such that $|x - c| < \varepsilon$. Intuitively, this says that x can be expressed as a "limit" of points in C (imagine making ε very very small, and smaller and smaller....). Notice that whatever c works for some ε_0 also works for all $\varepsilon > \varepsilon_0$, so it is only the "small" $\varepsilon > 0$ that are of interest here.

As a silly example, if C is an arbitrary subset of $\mathbb{R}$ then every element $x \in C$ is trivially an acculumation point of C (just take $c = x$ for any $\varepsilon > 0$). However, in the non-closed example $\{1/n \,|\, n \in \mathbb{N}\}$ we see that $x = 0$ is an accumulation point outside of the set. More generally:

Theorem 0.1. *Let $C \subseteq \mathbb{R}$. Then C is closed if and only if C coincides with its set of accumulation points (i.e., C is not a proper subset of its set of accumulation points).*

This makes the concept of closedness geometrically intuitive for drawing pictures, and makes precise the vague idea of "containing all endpoints". As you'll see, the proof is essentially just a matter of fiddling with definitions, but there is a certain geometric sensation that goes nicely with the idea of "containing all accumulation points" which is psychologically lacking in the description "complement is an open set". Thus, this fiddling with the definitions is worthwhile.

Proof. First assume that C is closed. We need to show that any $x \notin C$ is *not* an accumulation point of C. Well, $x \in \mathbb{R} - C$ and $\mathbb{R} - C$ is open, so for some $\varepsilon_x > 0$ we have $(x - \varepsilon_x, x + \varepsilon_x) \subseteq \mathbb{R} - C$, which is to say that this open ε_x-interval around x is *disjoint* from C. Thus, x cannot be an accumulation point (approximation of x by an element of C cannot be done to within an accuracy of less than this $\varepsilon_x > 0$).

Now assume that C coincides with its set of accumulation points. We must prove that C is closed, or equivalently that $\mathbb{R} - C$ is open. Pick $x \in \mathbb{R} - C$. We seek some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq \mathbb{R} - C$, or equivalently such that

$$(x - \varepsilon, x + \varepsilon) \cap C = \emptyset. \tag{1}$$

Why must some such $\varepsilon > 0$ exist? Well, $x \notin C$ and by hypothesis C coincides with its set of accumulation points, so x is *not* an accumulation point of C. But if we look back at the definition of "accumulation point", this says *exactly* that there must be *some* $\varepsilon > 0$ for which (1) holds. Indeed, if there were no such ε then for *every* $\varepsilon > 0$ the condition (1) would fail and hence we'd have exactly the condition that x *is* an accumulation point of C, yet $x \notin C$ so this contradicts the hypothesis that C coincides with its set of its accumulation points.

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As a mental exercise, note that $\emptyset$ is closed in $\mathbb{R}$, and does indeed coincide with its set of acccumulation points. It doesn't have any! Indeed, if x were an accumulation point to $\emptyset$ then say for $\varepsilon = 1 > 0$ there would have to be some $c \in \emptyset$ such that $|x - c| < 1$, but of course that's ridiculous: there are *no* elements of $\emptyset$. You should examine the above proof of $(\Leftarrow)$ to see that it really does work perfectly fine if $C = \emptyset$. Of course, the proof is correct so it has to work, but it is instructive to follow the proof of $(\Leftarrow)$ for $C = \emptyset$ to see what it says in that case.