Math 295. Homework 1 (Due September 17)

(1) Let S be a set. Recall that $\mathcal{P}(S)$ denotes the power set of S, that is, the set of all subsets of S. Define a binary operation $\star$ on $\mathcal{P}(S)$ by

$$A \star B = \begin{cases} A \cup B & \text{provided that } A \cap B = \emptyset \\ S & \text{provided that } A \cap B \neq \emptyset \end{cases}$$

for subsets A and B of S.

  (a) Show that $\star$ is an associative, commutative binary operation on $\mathcal{P}(S)$. (Note, this is mostly an exercise in "being careful".)

  (b) Is there a $\star$-identity? (Answering "yes" or "no" will get you zero points on such questions. You must both answer the question and justify your answer.)

  (c) If the answer to the above question was "yes", then how many elements of $\mathcal{P}(S)$ have $\star$-inverses?

(2) Suppose (F, P) is an ordered field. Suppose $a, b, c \in F$ with $a \geq 0$ and $b \geq 0$.

  (a) Show that $c \in P$ if and only if $c^{-1} \in P$.

  (b) Show that $a = b$ if and only if $a^2 = b^2$. (Note, if $b = 0$, then we have $a = 0$ if and only if $a^2 = 0$.)

  (c) Show that $a > 0$ if and only if $a^2 > 0$.

  (d) For $b > 0$, show that $a > b$ if and only if $a^2 > b^2$.

  (e) Use the above to conclude:

$$a > b \text{ if and only if } a^2 > b^2$$

  and

$$a = b \text{ if and only if } a^2 = b^2.$$

  We have already used this fact in class.

(3) The point of this problem is to show that there may be more than one way to order a field (that is, the choice of P is not necessarily uniquely determined). See page 572 for more discussion of this example. Define

$$\mathbb{Q}(\sqrt{2}) := \{(a + b\sqrt{2}) \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$$

  (a) Show that with respect to the usual binary operations of addition and multiplication in $\mathbb{R}$, the set $\mathbb{Q}(\sqrt{2})$ is a field. Be very careful about how you justify the existence of multiplicative inverses.

  (b) Suppose $z = a + b\sqrt{2}$, $w = r + s\sqrt{2}$ are two elements of $\mathbb{Q}(\sqrt{2})$. Show that if $z = w$, then $a = r$ and $b = s$.

  (c) From the previous part of this problem, we can unambiguously define $\overline{z} = a - b\sqrt{2}$. The operation $w \mapsto \overline{w}$ on $\mathbb{Q}(\sqrt{2})$ is called conjugation. It has many wonderful properties; for example, show that

    (i) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

    (ii) $\overline{z} = z$ if and only if $z \in \mathbb{Q}$.

    (iii) Formulate and prove a statement describing how conjugation and addition interact.

  (d) Define

$$P := \{z \in \mathbb{Q}(\sqrt{2}) \mid \overline{z} > 0 \text{ in } \mathbb{R} \}.$$

  Show that (F, P) is an ordered field. Since $-\sqrt{2} \in P$, this order is different from the standard order structure.

Book Problems Chapter 28: 8 (Note: By convention, if we also assume that F has an order structure, then when $a \neq 0$ has two square roots, we call the positive square root $\sqrt{a}$.) Chapter 1: 3, 5 (iv) – (vii), 12 (iv) – (vi).

You may wish to think about Chapter 1: 25 and Chapter 28: 1,2. Although finite fields are extremely interesting, they are not our focus this semester (nor, for that matter, this year).