Math 295. Homework 10 (Due December 10)

(1) The Cauchy-Schwartz inequality. Suppose that f and g are integrable functions on $[a, b]$. Show that

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right) \cdot \left(\int_a^b g^2\right).$$

Hint: See problem 39 on page 278 of Spivak.

(2) Define $\tilde{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$. We will say that a subset C of $\tilde{\mathbb{Z}}$ is closed if either C is finite or $\infty \in C$. A subset of $\tilde{\mathbb{Z}}$ is open provided that its complement is closed.
   (a) Show that $\tilde{\mathbb{Z}}$ is a topological space. (Hint: Don't forgot that you have proven a characterization of topological spaces in terms of closed sets while preparing for Exam II.)
   (b) Using de Morgan's Laws, formulate (and prove!) a definition of compactness for any topological space in terms of closed sets.
   (c) Is $\tilde{\mathbb{Z}}$ compact?

(3) (a) Suppose $h\colon I \to \mathbb{R}$ is a function, where I is some interval in $\mathbb{R}$. If h is differentiable at some $y \in I$ for which $h'(y) > 0$, then show that there is a neighborhood V of y so that for all $t \in V \cap I \setminus \{y\}$ we have

$$\frac{h(t) - h(y)}{t - y} > 0.$$

   (b) Find a polynomial that is strictly increasing on $\mathbb{R}$, but for whom the first derivative in not always positive. Does this contradict the previous problem? Why or why not?

(4) Suppose $f\colon I \to \mathbb{R}$ is a function on some interval $I \subset \mathbb{R}$. We say that f is Lipschitz provided that there exists a number C such that for all $x, y \in I$ we have

$$|f(x) - f(t)| \leq C\,|x - t|\,.$$

   (a) Prove that a Lipschitz function is continuous.
   (b) Prove that a Lipschitz function is uniformly continuous.
   (c) Prove that the function $h\colon [0, 1] \to \mathbb{R}$ that sends t to $\sqrt{t}$ is uniformly continuous, but not Lipschitz.
   (d) If $h\colon I \to \mathbb{R}$ is Lipschitz, must h be differentiable? What about the converse?

Book Problems Chapter 9: 13, 14 **Chapter 10**: 6, 18, 29, 31 **Chapter 11**: 62 **Chapter 14**: 1,3;