Math 295. Homework 3 (Due October 1)

(1) Let S be a set. A function $f\colon S \to S$ is called an involution provided that $f \circ f(s) = s$ for all $s \in S$. Show: If $g\colon S \to S$ is an involution, then g is bijective.

(2) Suppose $a, b \in \mathbb{R}$ with $a < b$. Show that there exist $c, d \in \mathbb{R}$ with $c \in \mathbb{Q}$ and $d \notin \mathbb{Q}$ so that $a < c < d < b$.

(3) Show that every nonempty bounded above subset of $\mathbb{N}$ has a maximal element. That is, if $S \subset \mathbb{N}$ is bounded above, then there is an element $t \in S$ so that $t \geq s$ for all $s \in S$.

(4) Fun with maps.

 (a) Suppose A is a set and $f\colon \mathbb{N} \to A$ is an injection. Show that there is a bijection between A and a proper subset of A. (That is, there is a subset A' of A and a map $g\colon A \to A'$ such that $A' \neq A$ and g is bijective.)

 (b) For $n \in \mathbb{N}$, let $\mathbb{N}_n := \{k \in \mathbb{N}\colon k \leq n\}$. Suppose $\ell, m \in \mathbb{N}$. Show that $\mathbb{N}_\ell = \mathbb{N}_m$ if and only if $\ell = m$. Serious hint: One direction is easy; for the other direction, use induction on $\max(\ell, m)$.

 (c) Suppose $n \in N$. Show that if S is a nonempty subset of $\mathbb{N}_n$, then there is a bijective map from S to $\mathbb{N}_\ell$ for some $\ell \leq n$. Hint: use induction.

 (d) Compare your answers to parts (4a) and (4c).

(5) What is a finite set anyway?

 (a) A set A is said to be finite if there is an $n \in \mathbb{N}$ and a bijective function $f\colon \mathbb{N}_n \to A$. Conclude from the previous problem that n is uniquely determined by A. We say that A has n elements.

 (b) Show that every finite subset of an ordered field has a maximal element.

 (c) Show that every finite subset of an ordered field has a minimal element.

(6) Suppose A, B, and C are sets. Suppose $f\colon A \to B$ and $g\colon B \to C$. Under what conditions on f and g will $g \circ f$ be

 (a) injective?

 (b) surjective?

 (c) bijective?

Suppose f^{-1} and g^{-1} are functions. Is the inverse of $g \circ f$ a function?

Book Problems. Chapter 1: 20; Chapter 4: 14 (i) - (vii), Chapter 12: 2,3,4 (the book defines increasing, decreasing)

Just for fun. Don't even read the statement of this problem until you have completed every other problem. Show that there does not exist an equilateral triangle in $\mathbb{R}^2$ whose vertices all lie in $\mathbb{Z}^2$.