Math 295. Homework 4 (Due October 8)

(1) **Countable Sets.**
   (a) Prove that a union of finitely many countable sets is countable.
   (b) Prove that a non-empty subset of a countable set is either finite or countable.
   (c) Prove that if A and B are countable, then so is $A \times B$.

(2) **Bounded Functions.** Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function. For $A \subset \mathbb{R}$ say that f is bounded on A provided that there is an $N \in \mathbb{R}$ such that $|f(x)| < N$ for all $x \in A$. We say that f is locally bounded if for every $x \in \mathbb{R}$ there is a neighborhood U of x for which f is bounded on U.
   (a) Show that if $h\colon \mathbb{R} \to \mathbb{R}$ is bounded on $\mathbb{R}$, then it is locally bounded.
   (b) Show that the function $\ell\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\ell(x) = \begin{cases} 0 & x = 0 \\ 1/x & x \neq 0 \end{cases}$$

   is not bounded on $\mathbb{R}$.
   (c) Show that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ n & x \in \mathbb{Q} \text{ and } x = m/n \text{ in lowest terms} \end{cases}$$

   is not locally bounded. (Remark. By convention, $0 = 0/1$ in lowest terms.)
   (d) Show that $r\colon \mathbb{R} \to \mathbb{R}$ defined by $r(x) = x^2$ is locally bounded but not bounded on $\mathbb{R}$.

(3) **Definition of exponentiation.** Suppose $a, b \in \mathbb{R}$ are both positive.
   (a) Show that $a < b$ implies $a^n < b^n$ for all $n \in \mathbb{N}$.
   (b) Show that if $a^n < b^n$ for some $n \in \mathbb{N}$, then $a < b$.
   (The notation a^n is defined inductively as follows: $a^1 = a$, and $a^{n+1} = a^n \cdot a$ for all $n \geq 1$.).

Book Problems. Chapter 2: 19, 23; Chapter 3: 5 (iv) – (viii), 8, 9, 11, 12, 13, 16

Just for fun. Don't even read the statement of this problem until you have completed every other problem. Show that for all

$$\sum_{j=2}^{n} \frac{1}{j}$$

is never an integer for $n \geq 2$.