Math 295. Homework 7 (Due November 5)

(1) **Topological Space structures on finite sets.**
   (a) Consider the set $X = \{a, b, c\}$ consisting of three elements. How many *different* topologies can be defined on this set?
   (b) We say that two topological spaces X and Y are *homeomorphic* if there is a *bijection* between them under which the open sets of X correspond precisely to the open sets of Y.[1] If your answer to (a) was n, let $X_1, \ldots, X_n$ be the n different topological spaces you found in (1). Which of these are homeomorphic to each other? How many "truly different" (non-homeomorphic) topologies are there on a three-point set; more precisely, how many three-point topological spaces are there, up to homeomorphism.

(2) **Non-Standard Topologies on the real line.** There are many different notions of "open" for $\mathbb{R}$. We could, in a fit of madness, decide to declare that the open subsets of $\mathbb{R}$ are the sets of the form
$$(a, \infty),$$
as a ranges over all the real numbers, together with the sets $\emptyset$ and $\mathbb{R}$.
   (a) Show that this forms a topology on $\mathbb{R}$.
   (b) Find a function which is continuous on $\mathbb{R}$ with respect to the usual Euclidean topology but not this one. Is it possible to find a function which is continuous with respect to this one but not the usual Euclidean topology?

(3) **Maps and subspace topology.** Suppose $X \xrightarrow{f} Y$ is a continuous map between topological spaces.
   (a) For any subset $A \subset X$, the restriction of f to A is the function $A \xrightarrow{f_{|A}} Y$ sending $a \in A$ to $f(a)$. Show that $f_{|A}$ is continuous, where $A \subset X$ is given the subspace topology.
   (b) Show that the function $f\colon X \to f(X)$ is continuous, where $f(X) \subset Y$ is given the subspace topology.
   (c) Let $f : [a, b] \to \mathbb{R}$ be a function where $[a, b]$ is a closed interval in $\mathbb{R}$. Show that f is continuous (in the abstract sense of topological spaces) if and only if f is continuous on (a, b) *and* f is continuous from the left at a and from the right at b in the $\delta - \varepsilon$ sense.[2]

(4) Show that if X has the discrete topology, then *any* map $X \to Y$ **to** *any* topological space is continuous. Show also if Y has the trivial topology, then *any* map $X \to Y$ **from** *any* topological space is continuous.

(5) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ 1/n & x \in \mathbb{Q} \text{ and } x = m/n \text{ in lowest terms} \end{cases}$$
Show that f is continuous at each irrational number and not continuous at each rational number.

(6) Show that the union of a finite number of compact sets (in an arbitrary topological space) is compact. Is the result true if we remove the "finite" condition? What about intersections?

Book Problems Chapter 5: 37a; Chapter 6: 6b; Chapter 7: 5, 11 (this is called the Brauer fixed point theorem in one-dimension); Chapter 8: 6.

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[1]Note: homeomorphism is word for "the same after relabling" in the world of topology, just as isomorphism is the word for "the same after relabling" in the world of fields discussed in September.

[2]Left and Right Continuity was defined on the Exam.