Math 295. Homework 8 (Due November 12)

(1) Suppose $f\colon X \to \mathbb{R}$ is a function, where X is a topological space. We say that f is locally constant provided that for all $x \in X$ there exists a neighborhood U (meaning an open set U containing x) and $d \in \mathbb{R}$ such that $f(y) = d$ for all $y \in U$. Prove that:
   (a) If X is connected and $f\colon X \to \mathbb{R}$ is continuous and locally constant, then f is a constant function.
   (b) If every continuous, locally constant function $f\colon X \to \mathbb{R}$ is constant, then X is connected.

(2) Let $X \to \mathbb{R}$ be a continuous function from an arbitrary *compact* topological space to $\mathbb{R}$ (with the Euclidean topology). Prove that f is not surjective. What more can you say about the image of f?

(3) Fix any topological space X and a point $x \in X$. We define a new topology, called the x-pointed space at X as follows: the open sets are (the empty set and) those sets which are both *open in* X and *contain* x.
   (a) Prove that this is really a topology on X.
   (b) Draw some pictures of open sets in this new topology in the case $X = \mathbb{R}$ and $x = 0$.
   (c) Find (with proof) an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous in the Euclidean topology but not in the 0-pointed topology on $\mathbb{R}$.
   (d) Can you find (with proof) an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous in the 0-pointed topology on $\mathbb{R}$, but not in the Euclidean topology?
   (e) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function in the Euclidean topology. Find and prove a necessary and sufficient condition for f to also be open in the 0-pointed topology.
   (f) Prove or disprove: the set $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is compact in the 0-pointed topology on $\mathbb{R}$.

(4) Let X be a topological space. The closure of a set S is the intersection of all closed sets containing S.
   (a) Show that the closure of S is the smallest closed set containing S. (This means that any other closed set T containing S must contain the closure of S).
   (b) Find (with proof) the closure of the bounded open interval (a, b) in $\mathbb{R}$ (with the usual Euclidean topology).
   (c) Find (with proof) the closure of the rational numbers in $\mathbb{R}$ (with the usual Euclidean topology).
   (d) Show that the closure of S is all of X if and and only if S intersects every non-empty open set of X. (In this case, we say S is dense in X).
   (e) Find an example of a countable dense subset of $\mathbb{R}$.

(5) Find a non-trivial topology on a set of three points in which the closure of every non-empty open set is the whole space.[1]

(6) Suppose that $f\colon I \to \mathbb{R}$ is a function where I is some interval in $\mathbb{R}$. The function f is said to be *strictly increasing on* I provided that for all $x, y \in I$, if $x < y$, then $f(x) < f(y)$. Similarly, the function f is said to be *strictly decreasing on* I provided that for all $x, y \in I$, if $x < y$, then $f(x) > f(y)$. Show that if $g\colon I \to \mathbb{R}$ is injective and continuous, then g is either strictly increasing or strictly decreasing.

(7) Show that if $h : [a, b] \to \mathbb{R}$ is a continuous injective map, then $h^{-1}\colon h([a, b]) \to [a, b]$ is continuous. (That is, a continuous injective map from $[a, b]$ to $\mathbb{R}$ is a homeomorphism onto its image.)

Book Problems Chapter 8 (appendix on p.144): 1,2,4

[1]Topological spaces like this come up a lot in number theory and algebra. For example, the so-called spectrum of any two-dimensional valuation ring is a isomorphic to a space like this.