Math 295. Homework 8 (Due November 12)

(1) Suppose \( f : X \to \mathbb{R} \) is a function, where \( X \) is a topological space. We say that \( f \) is locally constant provided that for all \( x \in X \) there exists a neighborhood \( U \) (meaning an open set \( U \) containing \( x \)) and \( d \in \mathbb{R} \) such that \( f(y) = d \) for all \( y \in U \). Prove that:
   (a) If \( X \) is connected and \( f : X \to \mathbb{R} \) is continuous and locally constant, then \( f \) is a constant function.
   (b) If every continuous, locally constant function \( f : X \to \mathbb{R} \) is constant, then \( X \) is connected.

(2) Let \( X \to \mathbb{R} \) be a continuous function from an arbitrary compact topological space to \( \mathbb{R} \) (with the Euclidean topology). Prove that \( f \) is not surjective. What more can you say about the image of \( f \)?

(3) Fix any topological space \( X \) and a point \( x \in X \). We define a new topology, called the \( x \)-pointed space at \( X \) as follows: the open sets are (the empty set and) those sets which are both open in \( X \) and contain \( x \).
   (a) Prove that this is really a topology on \( X \).
   (b) Draw some pictures of open sets in this new topology in the case \( X = \mathbb{R} \) and \( x = 0 \).
   (c) Find (with proof) an example of a function \( f : \mathbb{R} \to \mathbb{R} \) which is continuous in the Euclidean topology but not in the \( 0 \)-pointed topology on \( \mathbb{R} \).
   (d) Can you find (with proof) an example of a function \( f : \mathbb{R} \to \mathbb{R} \) which is continuous in the \( 0 \)-pointed topology on \( \mathbb{R} \), but not in the Euclidean topology?
   (e) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function in the Euclidean topology. Find and prove a necessary and sufficient condition for \( f \) to also be open in the \( 0 \)-pointed topology.
   (f) Prove or disprove: the set \( S = \{ \frac{1}{n} | n \in \mathbb{N} \} \) is compact in the \( 0 \)-pointed topology on \( \mathbb{R} \).

(4) Let \( X \) be a topological space. The closure of a set \( S \) is the intersection of all closed sets containing \( S \).
   (a) Show that the closure of \( S \) is the smallest closed set containing \( S \). (This means that any other closed set \( T \) containing \( S \) must contain the closure of \( S \)).
   (b) Find (with proof) the closure of the bounded open interval \((a, b)\) in \( \mathbb{R} \) (with the usual Euclidean topology).
   (c) Find (with proof) the closure of the rational numbers in \( \mathbb{R} \) (with the usual Euclidean topology).
   (d) Show that the closure of \( S \) is all of \( X \) if and only if \( S \) intersects every non-empty open set of \( X \). (In this case, we say \( S \) is dense in \( X \)).
   (e) Find an example of a countable dense subset of \( \mathbb{R} \).

(5) Find a non-trivial topology on a set of three points in which the closure of every non-empty open set is the whole space.\(^1\)

(6) Suppose that \( f : I \to \mathbb{R} \) is a function where \( I \) is some interval in \( \mathbb{R} \). The function \( f \) is said to be strictly increasing on \( I \) provided that for all \( x, y \in I \), if \( x < y \), then \( f(x) < f(y) \). Similarly, the function \( f \) is said to be strictly decreasing on \( I \) provided that for all \( x, y \in I \), if \( x < y \), then \( f(x) > f(y) \). Show that if \( g : I \to \mathbb{R} \) is injective and continuous, then \( g \) is either strictly increasing or strictly decreasing.

(7) Show that if \( h : [a, b] \to \mathbb{R} \) is a continuous injective map, then \( h^{-1} : h([a, b]) \to [a, b] \) is continuous. (That is, a continuous injective map from \([a, b]\) to \( \mathbb{R} \) is a homeomorphism onto its image.)

\textbf{Book Problems} Chapter 8 (appendix on p.144): 1,2,4

---

\(^1\)Topological spaces like this come up a lot in number theory and algebra. For example, the so-called spectrum of any two-dimensional valuation ring is an isomorphic to a space like this.