Math 295. Homework 9 (Due December 3)

(1) Suppose $a < b$. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f is integrable.
   (a) While you are away, somebody changes the function f to a function g. The functions f and g agree except at one million points. Is g integrable? Prove your answer.
   (b) Suppose f is non-negative and that for each $\varepsilon > 0$ the set $\{x \in [a,b] : f(x) > \varepsilon\}$ is finite. Show that $\int_a^b f = 0$.

(2) Suppose that I is an open interval and that $f\colon I \to \mathbb{R}$ is twice differentiable. Prove that if there are three distinct points in I where f is zero, then there is a point in I where f'' is zero.

(3) On page 265 of the third edition of our text, the author states:

   With these definitions, the equation $\int_a^c f + \int_c^b f = \int_a^b f$ holds for all a, c, b even if $a < c < b$ is not true (the proof of this assertion is a rather tedious case-by-case check).

   The proof is not that tedious. Do it.

(4) Define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \exp(-x^{-2}) & \text{otherwise.} \end{cases}$$

   Show that g is infinitely differentiable at 0, and, in fact, the k^{th} derivative of g at zero is 0. [Here, $exp(y)$ means e^y, the usual natural exponential; you may assume what you learned in high school regarding it's derivative.]

(5) Suppose $\mathcal{C} = \{C_1, C_2, C_3, \ldots\}$ is a set of nonempty, bounded, nested, closed intervals of $\mathbb{R}$. (So, each C_i is nonempty, closed, bounded, and $C_1 \supset C_2 \supset C_3 \supset \cdots$.)
   (a) Show that the intersection over all elements of $\mathcal{C}$ is nonempty. (Hint: Compactness.)
   (b) Show, by example, that each adjective (nonempty, bounded, nested, closed) modifying the word "interval" is necessary. For example, one of the four things you must do in this part of the problem is produce a collection of nonempty, unbounded, nested, closed intervals of $\mathbb{R}$ whose intersection is empty.

(6) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has the following property:

$$|f| \text{ is differentiable at } 0.$$

   Show, by example, that f need not be differentiable at zero. Show that if we also require that f is continuous at 0, then f is differentiable at 0.

Book Problems Chapter 13: 23(a)(b), 31(a)(b), 38
Chapter 11: 51, 52;