Math 295. A solution set for Homework 2

(1) Suppose that $A$ and $B$ are nonempty subsets of $\mathbb{R}$. Suppose that for all $a \in A$ and for all $b \in B$ we have $a < b$.

(a) Show that $\sup(A) \leq \inf(B)$.

We begin by showing that $\sup(A)$ and $\inf(B)$ exist. Choose $b \in B$. Since for all $a \in A$ we have $a < b$, we conclude that $A$ is bounded above by $b$. Since $A$ is also nonempty, we conclude that $\sup(A)$ exists. Now choose $a \in A$. Since $a < b$ for all $b \in B$, we conclude that $B$ is bounded below by $a$. Since $B$ is also nonempty, we conclude that $\inf(B)$ exists.

Since every element of $B$ is an upperbound of $A$, we conclude that, as $\sup(A)$ is the least upper bound, we must have $\sup(A) \leq b$ for all $b \in B$. But then $\sup(A)$ is a lower bound for $B$. Thus, as $\inf(B)$ is the greatest lower bound of $B$, we must have $\sup(A) \leq \inf(B)$.

(b) Show that $\sup(A) = \inf(B)$ if and only if for all $n \in \mathbb{N}$ there exist $a \in A$ and $b \in B$ so that $b - a < 1/n$.

⇒ We suppose that $\sup(A) = \inf(B)$. Choose $n \in \mathbb{N}$. We need to show that there is an $a \in A$ and a $b \in B$ so that $b - a < 1/n$. Since $\sup(A)$ is the least upper bound of $A$, there must be an element $a \in A$ so that $\sup(A) - 1/(2n) < a \leq \sup(A)$.

(If no such $a$ existed, then $\sup(A) - 1/(2n)$ would be an upper bound for $A$, contradicting the fact that $\sup(A)$ is the least upper bound of $A$.) Similarly, there is a $b \in B$ so that $\inf(B) \leq b < \inf(B) + 1/(2n)$.

Hence, for the elements $a$ and $b$ just chosen, we have

$$b - a = b - (\inf(B) - \sup(A)) - a$$

$$= (b - \inf(B)) + (\sup(A) - a) < 1/(2n) + 1/(2n) = 1/n.$$  

⇐ We shall prove this direction by establishing the contrapositive. That is, we shall suppose that $\inf(B) \neq \sup(A)$ and show that there is an $m \in \mathbb{N}$ so that $b - a > 1/m$ for all $a \in A$ and for all $b \in B$. If $\inf(B) \neq \sup(A)$, then from the previous problem we must have $\inf(B) < \sup(A)$. From the Archimedean property, we can choose $m \in \mathbb{N}$ so that $1/m < \sup(A) - \inf(B)$. Suppose $a \in A$ and $b \in B$. We have

$$b - a = b + (-a) \geq \inf(B) + (-\sup(A))$$

$$= \inf(B) - \sup(A) > 1/m.$$  

(c) Find an example for which $\sup(A) = \inf(B)$.

Consider the sets

$$A = \{-1/n : n \in \mathbb{N}\}$$

and

$$B = \{1/n : n \in \mathbb{N}\}.$$  

We want to show that $\sup(A) = \inf(B)$. It may be obvious to you that $0 = \sup(A) = \inf(B)$. However, I do not feel like proving this today. Instead, we note that if $m \in \mathbb{N}$, then for $a = -1/(2m + 2) \in A$ and $b = 1/(2m + 2)$ we have

$$b - a = 1/(2m + 2) + 1/(2m + 2) = 1/(m + 1).$$  

As $m + 1 > m$, we have $1/(m + 1) < 1/m$.

We have just shown that for all $m \in \mathbb{N}$ there is an $a \in A$ and a $b \in B$ so that $b - a < 1/m$. So, by the previous part of this problem, we conclude that $\sup(A) = \inf(B)$.

Remark This idea of proving things are the same without actually computing them is very handy. Keep it in mind.

(2) Suppose that $A$ is a nonempty bounded above subset of $\mathbb{R}$. Suppose $\sup(A) \notin A$. 
(a) Show that for all $n \in \mathbb{N}$ there are two elements (call them $a_1$ and $a_2$) of $A$ with the property that 

$$\sup(A) - 1/n < a_1 < a_2 < \sup(A).$$

Fix $n \in \mathbb{N}$.

Since $\sup(A)$ is the least upper bound of $A$, there is an $a_1 \in A$ so that 

$$\sup(A) - 1/n < a_1 \leq \sup(A).$$

Since $\sup(A) \notin A$, we conclude that $a_1 < \sup(A)$.

Since $\sup(A)$ is the least upper bound of $A$, there is an $a_2 \in A$ so that 

$$a_1 < a_2 \leq \sup(A).$$

Since $\sup(A) \notin A$, we conclude that $a_2 < \sup(A)$.

Thus, 

$$\sup(A) - 1/n < a_1 < a_2 < \sup(A).$$

Note, if we remove the assumption that $\sup(A) \notin A$, then the problem may be false. Consider the set $A = \{0\}$.

(b) If you have finished every other problem on this homework, prove that, in fact, there are infinitely many elements of $A$ between $\sup(A) - 1/n$ and $\sup(A)$.

I have not yet finished every other problem on this homework. Moreover, this problem is very hard to do without some basic understanding of what it means to be a finite set, etc. For this reason, I will do it after we do next weeks homework.

(3) An element of $\mathbb{R}$ is called irrational provided that it is not rational. That is, the set of irrationals is the set 

$$\{x \in \mathbb{R} : x \notin \mathbb{Q}\}.$$

(a) Is the sum of two irrationals always irrational?

No. We know that $\sqrt{2}$ is irrational. Moreover, $-\sqrt{2}$ is irrational as well. Yet the sum of these two numbers is $0$, a rational number.

(b) Is the product of two irrationals always irrational?

No. The product of $\sqrt{2}$ with itself is $2$, a rational number.

(c) Is the sum of a rational and an irrational always irrational? If not, when is it irrational?

Yes. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $q \in \mathbb{Q}$. We need to show that $\alpha + q \in \mathbb{R} \setminus \mathbb{Q}$. Suppose that this is not the case. Then there is a $p \in \mathbb{Q}$ so that 

$$\alpha + q = p.$$ 

But then $\alpha = p - q$; hence $\alpha$ is rational, a contradiction.

(d) Is the product of a rational and an irrational always irrational? If not, when is it irrational?

No, the rational number zero is the problem. For example $\sqrt{2} \cdot 0 = 0$.

**Book Problems Chapter 1: 14 (a):** If $a = 0$, then as $-0 = 0$, we have 

$$0 = |a| = |-a|.$$
Suppose \( a > 0 \), that is \( a \in P \). In this case, \( |a| = a \). On the other hand \( |-a| = -(a) \). However, we know that \(-(a)\) is \( a \). Thus \( |a| = |-a| \). Finally, we consider the case when \(-a \in P \). In this case \( |a| = -a \). Since \( |-a| = -a \), we conclude that \( |a| = |-a| \).

Chapter 1: 14 (b): “⇒”: If \(-b \leq a \leq b\), then \(-b \leq -a \leq b\). Hence, both \( a \) and \(-a\) are less than or equal to \( b \), so \( |a| \), which is either \( a \) or \(-a\), is less than or equal to \( b \).

“⇐”: Since \( 2 = 1 + 1 > 0 \), we know that \( |a| \geq -|a| \). Hence,

\[
-|a| \leq |a| \leq b.
\]

We therefore have both \(-a \leq b \) and \( a \leq b \). Consequently, we have both \(-b \leq a \) and \(-b \leq -a \). So, we conclude that \(-b \leq a \leq b \).

Chapter 1: 14 (c): From the above, to show \( |a + b| \leq |a| + |b| \) it is enough to show:

\[
-|a| - |b| \leq a + b \leq |a| + |b|.
\]

However, this follows immediately from the inequalities:

\[
-|a| \leq a \leq |a|\]

and

\[
-|b| \leq b \leq |b|.
\]

Chapter 1: 19 (b): We know \( 0 \leq (x - y)^2 = x^2 - 2xy + y^2 \). Hence \( 2xy \leq x^2 + y^2 \). Following the hint gives us:

\[
\frac{2x_1y_1}{\sqrt{x_1^2 + x_2^2\sqrt{y_1^2 + y_2^2}}} \leq \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2}
\]

and

\[
\frac{2x_2y_2}{\sqrt{x_1^2 + x_2^2\sqrt{y_1^2 + y_2^2}}} \leq \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2}.
\]

From these two inequalities we derive

\[
\frac{2x_1y_1 + 2x_2y_2}{\sqrt{x_1^2 + x_2^2\sqrt{y_1^2 + y_2^2}}} \leq 2.
\]

The desired result follows.

Chapter 1: 19 (c): We first establish the given equality. Expanding the right-hand side yields:

\[
x_1^2y_1^2 + x_2^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2
\]

(the inner terms like \(x_1y_1x_2y_2\) all cancel). Expanding the left-hand side yields:

\[
x_1^2y_1^2 + x_2^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2.
\]

So the two sides are equal.

Since \((x_1y_2 - x_2y_1)^2\) is positive, we conclude

\[
(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)
\]

from which the Schwartz inequality follows.

Chapter 2: 1(ii): Let \( S \) be the set of natural numbers \( k \) for which

\[
1^3 + 2^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2.
\]

We claim that \( S = \mathbb{N} \). Note that \( 1 \in S \). Suppose that \( m \in S \). If we can show that \( m + 1 \in S \), then we will have shown that \( S \) is inductive. Since \( \mathbb{N} \) is the smallest inductive subset of \( \mathbb{R} \) and \( S \subset \mathbb{N} \), we conclude that \( S = \mathbb{N} \).
Since \( m \in S \), we have
\[
1^3 + 2^3 + \cdots + (m + 1)^3 = 1^3 + 2^3 + \cdots + m^3 + (m + 1)^3 = (1 + 2 + \cdots + m)^2 + (m + 1)^3.
\]
From class (or book), this becomes
\[
(m(m + 1)/2)^2 + (m + 1)^3.
\]
Algebraic manipulation converts this into the form
\[
[(m + 1)(m + 2)/2]^2,
\]
which, again from class (or book), is
\[
(1 + 2 + \cdots + (m + 1))^2.
\]

Chapter 2: 3 (a): We assume \( n \geq k > 0 \). We have
\[
\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)! \cdot (n-k+1)!} + \frac{n!}{k! \cdot (n-k)!} = \frac{n!}{(k-1)! \cdot (n-k)!} \cdot \left( \frac{1}{n-k+1} + \frac{1}{k} \right) = \frac{n!}{(k-1)! \cdot (n-k)!} \cdot \left( \frac{n+1}{(n-k+1) \cdot k} \right) = \frac{n!}{k! \cdot (n+1-k)!} = \binom{n+1}{k}.
\]

Chapter 2: 3 (b): Let \( S \) be the set of natural numbers \( n \) for which \( \binom{n}{k} \) is a natural number for \( 0 \leq k \leq n \). Since \( \binom{1}{0} = 1 \) and \( \binom{1}{1} = 1 \), we have \( 1 \in S \). Suppose \( m \in S \). We will show \( m + 1 \in S \). Hence, \( S \) will be an inductive subset of \( \mathbb{N} \) and so we shall have \( S = \mathbb{N} \).

Suppose \( m \in S \). Note that \( \binom{m+1}{0} = 1 \) is a natural number. We also have that \( \binom{m+1}{1} = 1 \). Hence, we need to show that for \( 1 \leq k \leq m \) we have \( \binom{m+1}{k} \) is a natural number. From the previous part of this problem, we have
\[
\binom{m+1}{k} = \binom{m}{(k-1)} + \binom{m}{k}.
\]
Since \( m \in S \), we conclude that \( \binom{m+1}{k} \) is a natural number. Hence \( m + 1 \in S \).

Chapter 2: 3 (d): Let \( S \) be the set of natural numbers \( n \) for which
\[
(a + b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j.
\]
Clearly \( 1 \in S \). Suppose \( m \in S \). We shall show that \( (m + 1) \in S \) so that \( S \) is an inductive subset of \( \mathbb{N} \). As usual, this means that \( S = \mathbb{N} \).
We have, in glorious, tedious detail

\[(a + b)^{m+1} = (a + b) \cdot (a + b)^m\]

\[= (a + b) \cdot \sum_{j=0}^{m} \binom{m}{j} a^{m-j}b^j\]

\[= a \cdot \sum_{j=0}^{m} \binom{m}{j} a^{m-j}b^j + b \cdot \sum_{j=0}^{m} \binom{m}{j} a^{m-j}b^j\]

\[= \sum_{j=0}^{m} \binom{m}{j} a^{m+1-j}b^j + \sum_{j=0}^{m} \binom{m}{j} a^{m-j}b^{j+1}\]

\[= \sum_{j=0}^{m} \binom{m}{j} a^{m+1-j}b^j + \sum_{j=1}^{m+1} \binom{m}{j-1} a^{m+1-j}b^j\]

\[= \binom{m}{0} a^{m+1}b^0 + \sum_{j=1}^{m} \binom{m}{j} a^{(m+1)-j}b^j + \sum_{j=1}^{m} \binom{m}{j-1} a^{(m+1)-j}b^j + \binom{m}{m} a^0 b^{m+1}\]

\[= \binom{m+1}{0} a^{m+1}b^0 + \sum_{j=1}^{m+1} \left( \binom{m}{j} + \binom{m}{j-1} \right) a^{(m+1)-j}b^j + \binom{m+1}{m+1} a^0 b^{m+1}\]

\[= \sum_{j=0}^{m+1} \binom{m+1}{j} a^{(m+1)-j}b^j\]

Chapter 8: 13: See book.