Math 295. Solutions to Homework 4

(1) Coming soon

(2) Suppose \( f: \mathbb{R} \to \mathbb{R} \) is a function. For \( a, b \in \mathbb{R} \) with \( a < b \) we say that \( f \) is bounded on \((a, b)\) provided that there is an \( N \in \mathbb{R} \) such that \( |f(x)| < N \) for all \( x \in (a, b) \). We say that \( f \) is locally bounded if for every \( x \in \mathbb{R} \) there is a neighborhood \((a, b)\) of \( x \) for which \( f \) is bounded on \((a, b)\).

(a) Show that if \( h: \mathbb{R} \to \mathbb{R} \) is bounded (on \( \mathbb{R} \)), then it is locally bounded.

Since \( h: \mathbb{R} \to \mathbb{R} \) is bounded, there is an \( N \in \mathbb{R} \) such that \( |h(x)| < N \) for all \( x \in \mathbb{R} \). Choose \( y \in \mathbb{R} \). Suppose \((c, d)\) is a neighborhood of \( y \) in \( \mathbb{R} \). If \( w \in (c, d) \), then \( w \in \mathbb{R} \) and so \( |h(w)| < N \).

(b) Show that the function \( \ell: \mathbb{R} \to \mathbb{R} \) defined by
\[
\ell(x) = \begin{cases} 
0 & x = 0 \\
1/x & x \neq 0
\end{cases}
\]
is not bounded.

Suppose \( \ell: \mathbb{R} \to \mathbb{R} \) is bounded by \( N \). That is, for all \( y \in \mathbb{R} \) we have \( |\ell(y)| < N \). By the Archimedean property, we can find \( n \in \mathbb{N} \) so that \( n > N \). But then
\[
\ell(1/n) = n > N > \ell(1/n),
\]
a contradiction.

(c) Show that the function \( g: \mathbb{R} \to \mathbb{R} \) defined by
\[
g(x) = \begin{cases} 
0 & x \notin \mathbb{Q} \\
n & x \in \mathbb{Q} \text{ and } x = m/n \text{ in lowest terms}
\end{cases}
\]
is not locally bounded. (Remark, by convention, \( 0 = 0/1 \) in lowest terms.)

To show that \( g \) is not locally bounded, it is enough to show that it is not locally bounded near the origin. We need to show that for all \( N > 0 \) and for all neighborhoods \((a, b)\) of the origin, there is a \( y \in (a, b) \) so that \( |g(y)| > N \). Choose \( N > 0 \) and a neighborhood \((a, b)\) of the origin. Since \((a, b)\) is a neighborhood of the origin, there is an \( \varepsilon > 0 \) so that \( B_\varepsilon(0) \subset (a, b) \). By the Archimedean property, we can find \( n \in \mathbb{N} \) so that \( 1/n < \varepsilon \) and \( n > N \). Now \( g(1/n) = n > N \) and \( 1/n \in (a, b) \).

(3) Suppose \( a, b \in \mathbb{R} \) with \( a < b \). Suppose \( a, b \in \mathbb{R} \) are both positive.

(a) Show that \( a < b \) implies \( a^n < b^n \) for all \( n \in \mathbb{N} \).

A simple induction argument and the fact that \( a \) and \( b \) are positive shows us that \( a^n \) and \( b^n \) are positive for all \( n \in \mathbb{N} \).

Let
\[
S := \{ n \in \mathbb{N} : a^n < b^n \}.
\]
We shall show that \( S \) is weakly inductive. Clearly, \( 1 \in S \). Suppose \( n \in S \). We then have \( a^n < b^n \) and so
\[
a^{(n+1)} = aa^n < ba^n < bb^n = b^{(n+1)}.
\]

(b) Show that if \( a^n < b^n \) for some \( n \in \mathbb{N} \), then \( a < b \).

We will show that if \( a \geq b \), then \( a^n \geq b^n \) for all \( n \in \mathbb{N} \). This follows immediately from the previous problem.

Book Problems
Chapter 2: 19 See book.

Chapter 2: 22
(a) We start with the last part of the question: A calculation shows that if \( a_i = A_n \) for all \( i \) then \( G_n = A_n \). In what follows we show that if \( a_i \neq A_n \) for some \( i \), then \( G_n < A_n \). Thus, \( G_n = A_n \) if and only if \( a_i = A_n \) for all \( i \).

Suppose \( a_i \neq A_n \) for some \( i \). If \( a_i < A_n \) and then there is a \( j \) so that \( a_j > A_n \); otherwise, we have \( a_i < A_n \) and \( a_j \leq A_n \) for all \( j \) so that

\[
nA_n = a_1 + a_2 + \cdots + a_n < A_n + A_n + \cdots + A_n = nA_n
\]

so that \( A_n < A_n \), a contradiction. Similarly, if \( a_i > A_n \), then there is a \( j \) so that \( a_j < A_n \).

So, after rearranging the terms \( a_1, a_2, \ldots, a_n \), we may and do assume \( a_1 < A_n \) and \( a_2 > A_n \). We set \( \bar{a}_1 = A_n \), \( \bar{a}_2 = a_1 + a_2 - \bar{a}_1 \), and \( \bar{a}_j = a_j \) for \( j \geq 3 \). Let \( G_n^{(1)} \) denote the geometric mean of the \( \bar{a}_i \). We will show:

\[
G_n < G_n^{(1)}.
\]

Note that since \( a_1 < \bar{a}_1 < a_2 \), we have

\[
0 > (\bar{a}_1 - a_1)(\bar{a}_1 - a_2) = \bar{a}_1^2 - \bar{a}_1(a_1 + a_2) + a_1a_2 = -\bar{a}_1(a_2 + a_1 - \bar{a}_1) + a_1a_2 = -\bar{a}_1\bar{a}_2 + a_1a_2.
\]

Thus, \( a_1a_2 < \bar{a}_1\bar{a}_2 \). Consequently,

\[
(G_n)^n = a_1a_2a_3 \cdots a_n < \bar{a}_1\bar{a}_2a_3a_4 \cdots a_n = \bar{a}_1\bar{a}_2\bar{a}_3 \cdots \bar{a}_n = (G_n^{(1)})^n.
\]

To summarize: we have shown \( G_n < G_n^{(1)} \).

If each \( \bar{a}_i \) is equal to \( A_n \), then \( G_n^{(1)} = A_n \). Otherwise, we may modify the above argument as follows. After rearranging the terms of the sequence \( \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \), we may (and do) assume \( \bar{a}_1 = A_n, \bar{a}_2 < A_n \) and \( \bar{a}_3 > A_n \). We define \( \bar{a}_2 = A_n, \bar{a}_3 = \bar{a}_2 + \bar{a}_3 - \bar{a}_2, \) and \( \bar{a}_i = \bar{a}_i \) for all remaining \( i \). Note that \( \bar{a}_1 = \bar{a}_2 = A_n \). As before, we have

\[
\bar{a}_2 \bar{a}_3 > \bar{a}_2 \bar{a}_3
\]

and so

\[
(G_n^{(2)})^n := \bar{a}_1\bar{a}_2\bar{a}_3 \cdots \bar{a}_n > (G_n^{(1)})^n.
\]

Repeating this process, we end up with

\[
G_n < G_n^{(1)} < G_n^{(2)} < G_n^{(3)} \cdots.
\]

Eventually, we arrive at a \( k \) so that all of the terms contributing to \( G_n^{(k)} \) are \( A_n \). But, in this case, we have \( G_n^{(k)} = A_n \). So,

\[
G_n < G_n^{(1)} < G_n^{(2)} < G_n^{(3)} < \cdots < G_n^{(k)} = A_n.
\]

(b) We assume that \( G_2 \leq A_2 \) and \( G_{2k} \leq A_{2k} \). We then have

\[
G_{2(k+1)} = [a_1a_2 \cdots a_{2(k+1)}]^{1/2(k+1)}
= [(a_1a_2 \cdots a_{2k})(a_{2k+1}a_{2k+2} \cdots a_{2(k+1)})]^{1/2(k+1)}
= [(a_1a_2 \cdots a_{2k})^{1/2k}(a_{2k+1}a_{2k+2} \cdots a_{2(k+1)})^{1/2k}]^{1/2}
\]

Since \( G_2 \leq A_2 \),

\[
\leq \frac{(a_1a_2 \cdots a_{2k})^{1/2k} + (a_{2k+1}a_{2k+2} \cdots a_{2(k+1)})^{1/2k}}{2}
\]

Since \( G_{2k} \leq A_{2k} \),

\[
\leq \frac{a_1 + a_2 + \cdots a_{2k} + a_{2k+1} + a_{2k+2} + \cdots a_{2(k+1)}}{2k}
= A_{2k+1}
\]

(c) We apply part (b):
\[(G_n)^n A_n^{(2^m-n)} = a_1a_2 \cdots a_n A_n A_n \cdots A_n \]
\[\leq \left[ \frac{a_1 + a_2 + \cdots + a_n + (2^m - n)A_n}{2^m} \right]^{2^m} = \left[ \frac{nA_n + (2^m - n)A_n}{2^m} \right]^{2^m} = (A_n)^{2^m} \]

Thus \((G_n)^n \leq (A_n)^n\), which implies \(G_n \leq A_n\).

**Chapter 2: 23**

Fix \(n\).

We first show that \(a^{n+m} = a^n \cdot a^m\). Define

\[S := \{ m \in \mathbb{N} \mid a^{n+m} = a^n \cdot a^m \} \]

We need to show that \(S\) is weakly inductive. Since, by definition,

\[a^{\ell+1} = a^\ell \cdot a\]

for all \(\ell \in \mathbb{N}\), we have \(1 \in S\). Suppose \(k \in S\). We need to show \((k + 1) \in S\). Since \(k \in S\), we have

\[a^{n+k} = a^n \cdot a^k\]

Thus

\[a^{n+(k+1)} = a^{(n+k)+1} = a^{n+k} \cdot a = (a^n \cdot a^k) \cdot a = a^n \cdot (a^k \cdot a) = a^n \cdot (a^{k+1})\]

We now show that \(a^{nm} = (a^n)^m\). Define

\[S := \{ m \in \mathbb{N} \mid a^{nm} = (a^n)^m \} \]

We need to show that \(S\) is weakly inductive. Since, by definition,

\[a^{\ell+1} = a^\ell \cdot (a^\ell)^1\]

for all \(\ell \in \mathbb{N}\), we have \(1 \in S\). Suppose \(k \in S\). We need to show \((k + 1) \in S\). Since \(k \in S\), we have

\[a^{nk} = (a^n)^k\]

Thus

\[a^{n(k+1)} = a^{(nk)+n} = a^{nk} \cdot a^n = (a^n)^k \cdot (a^n)^1 = (a^n)^{k+1}\]

**Chapter 3: 5 (iv) – (viii)**

(iv) \(S \circ S\)

(v) \(P \circ P\)

(vii) \(s \circ (P + P \circ S)\)

(vii) \(s \circ s \circ s \circ P \circ P \circ P \circ s\)

(viii) \(P \circ S \circ s + s \circ S + P \circ s \circ (S + S)\)

**Chapter 3: 8**

In order to have \(f(f(x)) = x\) for all \(x\), we must have

\[0 = (ac + dc)x^2 + (d^2 - a^2)x - (ab + bd)\]

Thus, we require that \(0 = c(a + d), \ d = \pm a, \) and \(0 = b(a + d)\).

If \(d = a = 0\), then for \(f(x)\) and \(f(f(x))\) to make sense, we must have \(bc \neq 0\). So, in this case, we have: If \(d = a = 0\), then \(f(f(x)) = x\) as long as \(bc \neq 0\).

If \(d = a \neq 0\), then we must have \(b = c = 0\). So, in this case, we have: If \(d = a \neq 0\), then \(f(f(x)) = x\) as long as \(b = c = 0\).
If \( d = -a \neq 0 \), then \( f(x) \) makes sense as long as \( cx - a \neq 0 \); that is, \( x \neq a/c \). We also need to make sure that \( f(f(x)) \) makes sense, so we require that \( f(x) \neq a/c \); a little algebra shows that, under the assumption \( x \neq a/c \), this means that \( a^2 + bc \neq 0 \). So, the final answer in this case is: If \( d = -a \), then \( f(f(x)) = x \) as long as \( a^2 + bc \neq 0 \) and \( x \neq a/c \).

**Chapter 3: 9**

(a) We have

\[
C_{A \cap B} = C_A C_B ,
\]

\[
C_{A \cup B} = C_A + C_B - C_A C_B ,
\]

and

\[
C_{R \setminus A} = 1 - C_A .
\]

(b) Let \( A = \{ x \in \mathbb{R} \mid f(x) = 1 \} \).

(c) Suppose \( x \in \mathbb{R} \) and \( f(x)^2 = f(x) \). Then \( f(x)(f(x) - 1) = 0 \). We conclude that either \( f(x) = 0 \) or \( f(x) = 1 \). Set \( A = \{ y \in \mathbb{R} \mid f(y) = 1 \} \).

We have \( f = C_A \).

**Chapter 3: 11**

See book.

**Chapter 3: 12**

(a) The table for \( f + g \):

<table>
<thead>
<tr>
<th>( f + g )</th>
<th>( g ) odd</th>
<th>( g ) even</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f ) odd</td>
<td>( f ) odd</td>
<td>neither</td>
</tr>
<tr>
<td>( f ) even</td>
<td>neither</td>
<td>( f ) even</td>
</tr>
</tbody>
</table>

Note that the zero function is both even and odd. Thus to say \( f + g \) is even (resp. odd) for all \( f \) and for all \( g \) implies that \( f = f + 0 \) and \( g = g + 0 \) must be even (resp. odd). Consequently, in general, the sum of an odd and an even function can never be even (or odd). It is straightforward to verify that the sum of even (resp. odd) functions is even (resp. odd).

(b) The table for \( fg \):

<table>
<thead>
<tr>
<th>( fg )</th>
<th>( g ) odd</th>
<th>( g ) even</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f ) odd</td>
<td>( f ) odd</td>
<td>odd</td>
</tr>
<tr>
<td>( f ) even</td>
<td>odd</td>
<td>( f ) even</td>
</tr>
</tbody>
</table>

This is straightforward to verify.

(c) The table for \( f \circ g \):

<table>
<thead>
<tr>
<th>( f \circ g )</th>
<th>( g ) odd</th>
<th>( g ) even</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f ) odd</td>
<td>( f ) odd</td>
<td>even</td>
</tr>
<tr>
<td>( f ) even</td>
<td>even</td>
<td>( f ) even</td>
</tr>
</tbody>
</table>

This is straightforward to verify.

(d) Suppose \( f \) is even. Since \( f(-x) = f(x) \), we have \( f(x) = f(|x|) \). For each \( n \in \mathbb{N} \), define

\[
g_n(x) = \begin{cases} 
  f(x) & x \geq 0 \\
  x^n & x < 0 
\end{cases}
\]

For all \( x \in \mathbb{R} \), we have

\[
f(x) = f(|x|) = g_n(|x|).
\]
Chapter 3: 13

(a) Define, for \( x \in \mathbb{R} \),
\[
E(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad O(x) = \frac{f(x) - f(-x)}{2}.
\]
These functions have the desired properties.

(b) Suppose \( E: \mathbb{R} \to \mathbb{R}, E': \mathbb{R} \to \mathbb{R}, O: \mathbb{R} \to \mathbb{R}, O': \mathbb{R} \to \mathbb{R} \) are even functions and \( O: \mathbb{R} \to \mathbb{R}, O': \mathbb{R} \to \mathbb{R} \) are odd functions so that
\[
f = E + O \quad \text{and} \quad f = E' + O'.
\]

We wish to show that \( E = E' \) and \( O = O' \).

We have \( E - E' = O - O' \). Since \( E - E' \) is even and \( O - O' \) is odd, it is enough to show that the only function from \( \mathbb{R} \) to \( \mathbb{R} \) that is both even and odd is the zero function. Suppose \( g: \mathbb{R} \to \mathbb{R} \) is even and odd. We then have
\[
g(x) = g(-x) = -g(x)
\]
for all \( x \in \mathbb{R} \). By trichotomy, we conclude that \( g(x) = 0 \) for all \( x \in \mathbb{R} \).

Chapter 3: 16

(a) Proof by induction. Certainly \( f(x_1) = f(x_1) \). We suppose that \( f(x_1 + x_2 + \cdots + x_n) = f(x_1) + f(x_2) + \cdots + f(x_n) \).

We have
\[
f(x_1 + x_2 + \cdots + x_n + x_{n+1}) = f((x_1 + x_2 + \cdots + x_n) + x_{n+1})
\]
\[
f(x_1 + x_2 + \cdots + x_n) + f(x_{n+1})
\]
\[
= [f(x_1) + f(x_2) + \cdots + f(x_n)] + f(x_{n+1})
\]
\[
= f(x_1) + f(x_2) + \cdots + f(x_n) + f(x_{n+1})
\]

(b) We claim that if \( x \) is rational, then \( f(x) = f(1)x \).

Note that for \( n \in \mathbb{N} \) we have
\[
f(1) = f(n/n) = f(1/n) + f(1/n) + \cdots + f(1/n) = nf(1/n).
\]
Consequently, \( f(1/n) = f(1)/n \). For \( n, m \in \mathbb{N} \), we have
\[
f(m/n) = f(1/n) + f(1/n) + f(1/n) + \cdots + f(1/n) = mf(1/n) = f(1)(m/n).
\]

To extend to negative rational numbers, we note that
\[
f(0) = f(0 + 0) = f(0) + f(0)
\]
so that \( f(0) = 0 \). Hence, for all \( x \in \mathbb{R} \) we have
\[
0 = f(0) = f(x - x) = f(x) + f(-x).
\]
So \( f(-x) = -f(x) \).