Math 296. Problems for Final Exam

The Final Exam will be Thursday April 21 4–6 pm, in our usual classroom. In solving these problems, you may use all materials from your class notes, class website, and textbook. However, you may not seek help from other sources including the internet, your classmates, your friends, other texts, or any other unauthorized source. Seeking help from unauthorized sources is cheating. Cheaters will fail.

The Final Exam will have six problems, at least four of which will be taken directly from the following 12.

**Book Problems 8.2 # 4, 9, 13; Read also section 3.5 and do # 1, 6**

1. **From the 217 Final.** Let \( W \) be the plane in \( \mathbb{R}^3 \) defined by the equation \( x - z = 0 \).
   (1) Find an orthonormal basis for \( W \) (with respect to standard inner product).
   (2) Let \( R : \mathbb{R}^3 \to \mathbb{R}^3 \) be the reflection over the plane \( W \). Find the matrix of \( R \) with respect to the standard basis.
   (3) Diagonalize \( R \) by finding a basis \( B \) and a matrix such such \( R \) is diagonal when represented in this basis.
   (4) Is \( R \) invertible?

2. The trace of a square matrix is the sum of the entries on the main diagonal.
   (1) Show that the trace of \( AB \) is the trace of \( BA \) where \( A \) and \( B \) are square matrices of the same size.
   (2) Prove that similar matrices have the same trace.
   (3) Develop a definition of the trace of a linear transformation \( T : V \to V \) of a finite dimensional vector space. Explain why it is well-defined!
   (4) Describe the trace in terms of the characteristic polynomial of \( T \) and also in terms of the eigenvalues.

3. Let \( A \) be an \( n \times n \) matrix over a field \( F \).
   (1) Show that the map
   \[
   B : F^n \times F^n \to F \\
   (v, w) \mapsto v^T A w
   \]
   is bi-linear, and that every bilinear form \( F^n \times F^n \to F \) is represented by a matrix this way.
   (2) Show that \( B \) is symmetric if and only if \( A \) is symmetric.
   (3) In the case \( F \) is an ordered field, show that \( B \) is an inner product if and only if \( A = A^T \) and the subdeterminants formed by the first \( k \) columns and first \( k \) rows of \( A \) are all positive, for \( k = 1, \ldots, n \).

4. Show that all inner products on \( \mathbb{R}^n \) determine the same topology.

5. Let \( B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be a symmetric bilinear form. Show that there is a basis for \( \mathbb{R}^n \) in which the form \( B \) will be represented by a diagonal matrix with only \( \pm 1, 0 \) on the diagonal. Which of these represent the positive definite symmetric forms?

6. A Hermitian inner product on a complex vector space \( V \) is a map \( H : V \times V \to \mathbb{C} \), satisfying
   (1) \( H(\lambda v_1 + v_2, w) = \lambda H(v_1, w) + H(v_2, w) \) for all \( v_1, v_2, w \in V \) and all \( \lambda \in \mathbb{C} \).
   (2) \( H \) is conjugate symmetric: \( H(v, w) = \overline{H(w, v)} \) for all \( v, w \) in \( V \).
   (3) \( H(v, v) > 0 \) for all non-zero \( v \in V \).

Show that:
   (1) If \( H : V \times V \to \mathbb{C} \) satisfies (2), then \( H(v, v) \) is a real number, so (3) makes sense.
   (2) Show that \( H \) is not bilinear, but rather it is conjugate linear in the second argument. (Part of the problem is thus to figure out what is meant by conjugate linear.)
   (3) Show that \( \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \) sending \( ((z_1, \ldots, z_n), (w_1, \ldots, w_n)) \mapsto \sum_{i=1}^n z_i \bar{w}_i \) is a Hermitian inner product (called the standard Hermitian form).
   (4) Show that under the standard identification of \( \mathbb{C} \) with \( \mathbb{R}^2 \), the standard Hermitian form gives rise to the standard inner product on \( \mathbb{R}^2 \).
   (5) Show that a Hermitian inner product on a finite dimensional complex vector space \( V \) is determined by the values of \( H(v_i, v_j) \) for all pairs of vectors \( v_i \) and \( v_j \) in one fixed basis \( B = \{v_1, \ldots, v_n\} \).
(6) Show that every Hermitian inner product $H$ on a finite dimensional complex vector space is represented by an $n \times n$ matrix, which depends on the choice of a basis for $V$.

(7) Prove that the matrix you found represented $H$ is conjugate-symmetric: its entries satisfy $a_{ij} = \bar{a}_{ji}$.

7. Let $T : V \to W$ be a linear map of vector spaces. Let $T^* : W^* \to V^*$ be the naturally induced map of dual spaces, which sends $f \in W^*$ to the map $f \circ T$.

(1) Verify that $f \circ T$ really is an element of $V^*$.

(2) Prove that $T^*$ is a linear map.

(3) Show that $T$ is surjective if and only $T^*$ is injective.

(4) Show that $T$ is injective and only if $T^*$ is surjective.