1. **Dual Spaces.** Let $V$ be a vector space over $F$. Let $V^*$ be the dual space: By definition this is the space of linear maps $\phi : V \to F$.

(1) If $\{v_1, \ldots, v_n\}$ is a basis for $V$, show that $\{x_1, \ldots, x_n\}$ is a basis for $V^*$, where $x_i$ is the linear map sending $v_i$ to 1 and all other basis elements to 0. (This is called the “dual basis” corresponding to the basis $\{v_1, \ldots, v_n\}$.)

(2) Consider the map $\phi : V \to V^{**} = (V^*)^*$ which sends a vector $v$ in $V$ to the map $V^* \to F$ given by $\phi \mapsto \phi(v)$. Prove that this map is an isomorphism if $V$ is finite dimensional. This isomorphism is *natural*, meaning that it does not depend on a choice of basis.

(3) Prove also that $V \cong V^*$. (However, this is not a natural isomorphism!).

2. **From the 217 Final.** Let $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & -1 & 0 \\ 3 & 2 & 2 & 0 \\ -1 & 2 & 4 & 0 \end{pmatrix}$.

(1) Find the row reduced echelon form of $A$.

(2) Compute a basis for the kernel and image of $A$.

3. **Similar to one from the 217 Final, slightly harder.** Let $A$ be a $2 \times 2$ matrix and let $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be an eigenvector with eigenvalue $-2$ and let $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ be an eigenvector with eigenvalue $3$.

(1) Find the characteristic polynomial of $A$.

(2) Find a diagonal matrix $D$ similar to $A$, and the matrix $P$ such that $D = P^{-1}AP$.

4. **From 217 Final.** Give an example or explain why non exists.

(1) A $5 \times 5$ matrix whose rank and nullity (dim of kernel) are both 2.

(2) A linear transformation of $\mathbb{R}^2$ to itself with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(3) A surjective linear map $T : V_2 \to \mathbb{R}^3$ which satisfies $T(x) = 0$, where $V_2$ is the space of real polynomials (in one variable) of degree 2.

(4) A surjective linear map $T : \mathbb{R}^3 \to V_2$ which satisfies $T(e_1) = x^2 + x + 1$, where $V_2$ is the space of real polynomials of degree 2.

5. **From 217 Final.** Prove the following.

(1) Let $\lambda$ be an eigenvalue for $A$. Show that $\lambda^3 + \lambda^2 + 4$ is an eigenvalue for $A^3 + A + 4$.

(2) Show that if $v$ is an eigenvector for both $T$ and $S$, then $v$ is an eigenvector for $T \circ S$, where both $T$ and $S$ are linear transformations from $V$ to itself.

**Book Problems:** 3.6 # 1, 5.2 # 5, 5.3 # 9, 10 5.4 # 1, 2, 3, 4, 5, 7, 6.2 #1, 2, 4