1. **Equivalence Classes.** Let $R$ be an equivalence relation on a set $X$. For each $x \in X$, consider the subset $xR \subset X$ consisting of all the elements $y$ in $X$ such that $xRy$. A set of the form $xR$ is called an equivalence class.

   (1) Show that $xR = yR$ (as subsets of $X$) if and only if $xRy$.
   (2) Show that $xR \cap yR = \emptyset$ or $xR = yR$.
   (3) Show that there is a subset $Y$ (called equivalence classes representatives) of $X$ such that $X$ is the disjoint union of subsets of the form $yR$ for $y \in Y$. Is the set $Y$ uniquely determined?
   (4) For each of the equivalence relations from Problem Set 2, Exercise 5, Parts 3, 5, 6, 7, 8: describe the equivalence classes, find a way to enumerate them by picking a nice representative for each, and find the cardinality of the set of equivalence classes. [I will ask Ruthi to discuss this a bit in the discussion session.]

2. **Pliability of Smooth Functions.** This problem undertakes a very fundamental construction: to prove that $C^\infty$-functions are very soft and pliable. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = e^{-1/x^2}$ for $x \neq 0$ and $F(0) = 0$.

   (1) Verify that $F$ is infinitely differentiable at every point (don’t forget that you computed on a 295 problem set that the $k$-th derivative exists and is zero, for all $k \geq 1$).
   (2) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(x) = 0$ for $x \leq 0$ and $\varphi(x) = e^{-1/x^2}$ for $x > 0$. Show that $\varphi \in C^\infty(\mathbb{R})$.
   (3) Define $\psi(x) = \varphi(x)\varphi(1-x)$, and $\chi(x) = (\int_0^1 \psi)^{-1}\int_0^x \psi$. Show that $\chi$ is a monotone increasing function in $C^\infty(\mathbb{R})$ such that $\chi(x) = 0$ for $x \leq 0$ and $\chi(x) = 1$ for $x \geq 1$. [Hint: $\psi$ is positive for $x \in (0,1)$ and $\psi(x) = 0$ otherwise.]
   (4) Sketch graphs of $\chi$ and $1-\chi$.
   (5) For any two functions $f, g \in C^\infty(\mathbb{R})$, show that $(1-\chi)f + \chi g$ lies in $C^\infty(\mathbb{R})$ and is equal to $f$ on $(-\infty,0]$ and is equal to $g$ on $[1,\infty)$. Thus, we have “interpolated” from $f$ to $g$ without leaving the world of $C^\infty$-functions!

3. **More Pliability of Smooth Functions.** Let $I \subseteq \mathbb{R}$ be a (possibly unbounded) interval consisting of more than one point.

   (1) Prove that for any $a < a' < b < b'$ in $\mathbb{R}$ there exists $f \in C^\infty(\mathbb{R})$ such that $f(x) = 0$ for $x \notin [a,b]$, $f(x) = 1$ for $x \in [a',b']$, and $f$ is strictly increasing on $[a,a']$ and strictly decreasing on $[b',b]$; draw a picture of the graph. Using such an $f$ (often called a cutoff function for $[a',b']$ inside $[a,b]$) show that if $g \in C^\infty(I)$ and $[a,b]$ as above lies in $I$ then there exists $h \in C^\infty(\mathbb{R})$ such that $h = g$ on $[a',b']$ and $h = 0$ outside $(a,b)$; keep in mind that $h$ might blow up near endpoints of $I$, so $g$ need not extend to $C^\infty(\mathbb{R})$.
   (2) Let $L_1$ and $L_2$ be two linear functions which take the same value at a point $a \in \mathbb{R}$. Define a new, “piecewise linear” function $L$ by $L(x) = L_1(x)$ if $x \leq a$ and $L(x) = L_2(x)$ for $x \geq a$. Prove that $L$ is a uniform limit of $C^\infty$-functions.

4. **A topology problem, to keep us sharp.** A subset of $I \subseteq \mathbb{R}$ is said to be nowhere dense provided that for every open interval $(a,b)$ there is an open interval $(c,d) \subset (a,b)$ so that $(c,d) \cap I = \emptyset$.

   (1) Show that $\{1/n \mid n \in \mathbb{N}\}$ is nowhere dense.
   (2) (Baire) Show that an interval in $\mathbb{R}$ having more than one element cannot be written as the countable union of nowhere dense sets.

**Book problems.** Ch 22: 22 (for part (c) you should assume $0 \leq c < 1$ and you should get the conclusion under the slightly weaker assumption $|x_n - x_{n+1}| \leq Ae^a$ for all $n$ and a constant $A \geq 0$), 23 (assume $0 \leq c < 1$, and for part (c) you’ll find the proposed slight generalization of 22(c) above to be useful), 27 b (no justification needed), Ch 23: 7. Ch 24: 1(iii),(v), 2(iii),(v),(vii),

Exercises 22 and 23, together with 20 from last week, constitute a single chain of ideas that has profound consequences in the theory of differential equations and beyond.