Book Problems:

1.2: 6
1.3: 1, 2, 6
1.4: 4, 8
2.2: 8
3.1: 1

1. Groups. A group is a set $G$ together with one binary operation $\circ$ which is associative, has an identity (denoted $e$) and for which every element $g \in G$ has an inverse under $\circ$. For each set and binary operation below, decide whether or not the set is a group, and what the identity element is. Justify your answer.

1. Let $S_n$ be the set of all permutations (reorderings) of a set of $n$ objects. Let $\circ$ be composition. For example, the permutation switching the first two objects can be composed with the permutation switching the second two objects. This results in a permutation which moves the first three elements in a cycle (meaning the first goes to the third, the third goes to the second and the second goes to the first).

2. Fix a set $X$. Let $\text{Aut } X$ denote the set of all bijective self-maps (or automorphisms) of $X$. That is, $\text{Aut } X := \{f : X \to X \mid f \text{ is bijective}\}$. The operation $\circ$ is composition.

3. The set of integers $\mathbb{Z}$ under the operation of addition.

4. The set of integers under the operation of multiplication.

5. The set $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ of complex numbers of absolute value 1, under multiplication.

6. The set of $n$-th roots of unity (in $\mathbb{C}$) under multiplication (for some fixed $n$).

7. The set $F^\times$ of non-zero elements in a fixed field $F$, under the field multiplication.

8. The set $C^\infty(\mathbb{R})$ of smooth functions on $\mathbb{R}$, with the operation of composition.

9. The set of elements in a vector space $V$, under the vector addition.

2. A group $G$ is abelian if its operation $\circ$ is commutative. For each of the groups you identified in 1, tell which is abelian.

3. Let $I = [a, b]$ be a closed and bounded interval, and let $C^0(I)$ denote the space of continuous $\mathbb{R}$-valued functions on $I$. For $f \in C^0(I)$, define

$$|f| = \sup_{x \in I} |f(x)|.$$ 

1. Prove that $|f| \geq 0$ with equality if and only if $f = 0$, and that $|cf| = |c| \cdot |f|$ for $c \in \mathbb{R}$ and $f \in C^0(I)$.

2. Prove the triangle inequality $|f + g| \leq |f| + |g|$ for all $f, g \in C^0(I)$.

3. For $f, g \in C^0(I)$, define the uniform distance between $f$ and $g$ to be $d_u(f, g) = |f - g|$. Prove that $(C^0(I), d_u)$ is a metric space. In other words, show that $d_u$ is a metric (or distance) on $C^0(I)$. [Recall that a metric on a space $X$ is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ which satisfies the following three properties: 1) $d(x, y) = 0$ if and only if $x = y$; 2) $d(x, y) = d(y, x)$ (is symmetric); and 3) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).]

4. Say that a sequence $\{f_n\}$ in $C^0(I)$ is $d_u$-convergent to $f \in C^0(I)$ if and only if for all $\varepsilon > 0$ there exists $N_\varepsilon$ such that $d_u(f_n, f) < \varepsilon$ for all $n > N_\varepsilon$. Explain why $d_u$-convergence is exactly uniform convergence by another name.

5. A sequence $\{f_n\}$ is $d_u$-Cauchy if for all $\varepsilon > 0$ there exists $N_\varepsilon$ such that for all $n, m > N_\varepsilon$ we have $d_u(f_n, f_m) < \varepsilon$. Prove that any $d_u$-convergent sequence in $C^0(I)$ is $d_u$-Cauchy, and conversely (this is the key point) that any $d_u$-Cauchy sequence in $C^0(I)$ is convergent. We therefore say that the space $C^0(I)$ is complete with respect to the notion of $d_u$-distance.