1. Subgroups. A subgroup \( H \) of a group \( G \) is a subset, which is itself a group under the binary operation of \( G \).

(1) Prove that the identity element of \( H \) is the identity element of \( G \).

(2) Show that the set \( n\mathbb{Z} \) of multiplies of a fixed integer \( n \) is a subgroup of the group \((\mathbb{Z}, +)\) of integers under addition.

(3) Let \( H \) be a subgroup of \( \mathbb{Z} \) under addition. Show that if both \( n \) and \( m \) are in \( H \), then so is their greatest common divisor. [Hint: You might want to review Problem Set 5 from 295.]

(4) Describe all subgroups of \((\mathbb{Z}, +)\).

2. Symmetry Group of an Equilateral Triangle.

(1) Let \( D_3 \) be the group of symmetries of an equilateral triangle. This means, the group (under composition) of all rigid motions that cut the triangle out of the plane, and then replaces it so that it fits exactly. (We did a similar thing in class for the square). Prove that this group has exactly six members, and describe the group structure by recording the full table for the binary operation.

(2) Find all subgroups of \( D_3 \).

(3) Is \( D_3 \) abelian?

3. Generators for a group. We say that a group \((G, \circ)\) is generated by a set of elements \( S = \{g_i\}_{i \in I} \) if every element of \( G \) can be obtained as a product of elements (and their inverses) of the set \( S \). That is, if for all \( g \in G \), we can write \( g = g_1 \circ g_2 \cdots \circ g_t \) where each \( g_i \) or its inverse is in \( S \). Of course, many different sets of elements can generate a group; there is no uniqueness at all.

(1) Find two different sets of generators for the group of integers under addition.

(2) Find generators for the subgroups you found in 1 (4).

(3) Find generators for the group \( D_3 \).

(4) Show that any group generated by one element is abelian.

(5) Fix a set \( S \) of generators for a group \( G \). The word length with respect to \( S \) of an element \( g \) in \( G \) is the smallest \( t \) such that we can write \( g = g_1 \circ g_2 \cdots \circ g_t \) where each \( g_i \) or its inverse is in \( S \). Calculate the word length of each element in \( D_3 \) with respect to the generators you found in (3).

(6) The word length of a finite group \( G \) with respect to a fixed generating set \( S \) is the maximum of the word lengths with respect to \( S \) of all the elements in \( G \). What is the word length of \( D_3 \) with respect to the set of generators you found in (3).

(7) If you know how to solve the Rubik’s cube, estimate the word length of the Rubik’s cube group with respect to the six standard generators (quarter turn rotation of each face).

4. One Point Compactification. Suppose \( X \) is a Hausdorff topological space. The one-point compactification of \( X \), which we shall denote by \( \hat{X} \), is the topological space obtained by adding an extra point \( \infty_X \) (often called a point at infinity) to \( X \) (so \( \hat{X} = X \cup \{\infty_X\} \)) and defining the open sets of the new space to be the open sets of \( X \) together with the sets of the form \((X \setminus K) \cup \{\infty_X\}\), where \( K \) is a compact subset of \( X \).

(1) Show that \( \hat{X} \) is a topological space.

(2) Show that \( \hat{X} \) is compact.

(3) Think about the one-point-compactification of the real line (with the usual topology). Can you feel why this is the same as a circle? Can you prove that \( \hat{R} \) is homeomorphic to a circle (hint: this is a type of stereographic projection)?