Write your name on each sheet of paper.

This is a 50 minute closed book examination, with a point total of 100. Calculators are not permitted. Unjustified answers could receive no credit, even if they are correct. **Show your work.**

This exam contains one TF section worth 20 points, and three more problems whose point total is 80 points. For some problems, you will make choices about which problems you present. Only one will be graded. Please indicate precisely which problem you want me to grade.

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**WRITE OUT & SIGN PLEDGE:**

I pledge my honor that I have not used or sought mathematical assistance from any source apart from the course text and my class notes in connection with my work on this examination.
1. True or False. No explanation necessary.

(1) There exists a non-abelian group of order 49.
(2) A free abelian group on the set \( \{x, y\} \) is isomorphic to \( \mathbb{Z}^2 \).
(3) The tetrahedral group \( T \) (ie. the rotational symmetry group of a regular tetrahedron acts transitively on a set of order eight.
(4) In \( S_5 \), the elements \((123)(45)\) and \((245)(13)\) are conjugate to eachother.
(5) The group \( \mathbb{Z}_8 \times \mathbb{Z}_2 \) is solvable.
(6) The icosahedral group (ie, the rotational symmetry group of a regular dodecahedron or icosahedron) contains a subgroup of order 4.
(7) The tautological action of \( D_6 \) on \( \mathbb{R}^2 \) induces an action of \( D_6 \) on the subset of \( \mathbb{R}^2 \) consisting of points with integer coordinates.
(8) When \( O_3(\mathbb{R}) \) acts tautologically on \( \mathbb{R}^3 \), there is an induced action on the set of all lines though the origin in \( \mathbb{R}^3 \).
(9) The (additive) subgroup \( \mathbb{Z}^2 \) of \( \mathbb{R}^2 \) is a discrete group.
(10) If \( H_1 \) and \( H_2 \) are subgroups of \( G \) which are conjugate to each other under the conjugation action of \( G \) on itself, then \( H_1 \) and \( H_2 \) are isomorphic.
2. [25 points] Consider the group $G \subset GL_2(\mathbb{F}_p)$ of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$.

Describe explicitly a subgroup of order $p$ and show that it is the only one.
3. Solvable groups. [25 points]
   (1) Define what it means for a group $G$ to be solvable.
   (2) Prove, using Sylow’s theorems, that every group of order 12 is solvable.
4. Class Equation. [30 points]

Carefully state the class equation, defining any words you use it in. Explain what it has to do with groups acting on sets and orbits. Then **CHOOSE** one of the following:

**Choice 1.** Explain the class equation for each of the groups $D_6$ and $S_4$. For each, you should have an equation involving actual numbers, and you should tell me exactly what each of the numbers in your equation represents and/or counts. You **do not** have to prove that these numbers count the things you claim the count, as long as you describe what you are counting exactly.

**OR**

**Choice 2.** Use the class equation to prove that any group of prime power order has non-trivial center.