1. The Icosahedral Group. Let $G$ be the group of symmetries of the oriented icosahedron (see the picture on page 184: this is a solid figure formed by twenty equilateral triangles).

   (1) Show that $G$ can be identified with the group of symmetries of the oriented dodecahedron (a solid figure formed by twelve regular pentagons).
   (2) Find sets of cardinality 20, 30, 12, on which $G$ acts transitively (Hint: think of faces, vertices and edges).
   (3) Show that $G$ has order 60 by taking advantage of the actions you looked at in (2).

2. Discrete Subgroups of $\mathbb{R}^n$. Let $G$ be a subgroup of the (additive) group $\mathbb{R}^n$. We say a $G$ is discrete if it is discrete considered as subset of $\mathbb{R}^n$ under the usual Euclidean topology.¹

   (1) Show that $\mathbb{Z}^2$ is a discrete subgroup of $\mathbb{R}^2$. What about $\mathbb{Q}^2$?
   (2) Show that $G$ is a discrete subgroup of $\mathbb{R}^n$ if and only if there exists an $\epsilon$ such that all non-identity elements of $G$ are a distance greater than $\epsilon$ from the origin.
   (3) Prove that if $v_1$ and $v_2$ are linearly independent vectors in $\mathbb{R}^2$, then they generate a discrete subgroup of $\mathbb{R}^2$.
   (4) Draw a representative set of points in the group generated by $(1,0)$ and $(0,1)$, and another picture depicting the group generated by $(1,0)$ and $(1,\pi)$.
   (5) Is the subgroup generated by $(1,0),(0,1),(1,\pi)$ discrete?

3. Discrete Subgroups of $\mathbb{R}$. 

   (1) Show that a finite collection of real numbers $\{x_1,\ldots,x_d\}$ generates a discrete subgroup of $\mathbb{R}$ if and only if the elements $x_i$ span a one-dimensional $\mathbb{Q}$ vector subspace of $\mathbb{R}$.
   (2) Show that every discrete subgroup of $\mathbb{R}$ is isomorphic to $\mathbb{Z}$.

4. Discrete groups of Isometries. We say a subgroup $G$ of the group $M_2$ of isometries of $\mathbb{R}^2$ is discrete if there exists an $\epsilon > 0$ such that both of the following conditions hold: (1) Whenever a translation $t_a \in G$, $a > \epsilon$; and (2) Whenever a rotation $\rho_{\theta}$ through $\theta$ around some point $p$ is in $G$, then $\theta > \epsilon$.

   (1) Prove that every finite group of isometries of the plane is discrete.
   (2) Which of the following are discrete groups of isometries of the plane?
      (a) $SO_n(\mathbb{R})$,
      (b) The group of rotations around a fixed point $p$ through all rational multiples of $2\pi$.
      (c) The group generated by the translations $t_{(1,0)}$ and $t_{(0,1)}$.
      (d) The group generated by the translations $t_{(\pi,0)}$ and $t_{(0,\sqrt{2})}$.
      (e) The group generated by the translations $t_{(1,0)}, t_{(0,1)},$ and $t_{(\sqrt{2},0)}$.

5. The translation group of a discrete group. Let $G$ be a discrete group of isometries on $\mathbb{R}^2$. By definition, its translation group $T$ is the subgroup of all translations in $G$. Prove that the orbit of the origin under $T$ has a natural group structure which identifies it a discrete subset of $\mathbb{R}^2$ isomorphic to $T$.

¹This means that if for all $x \in X$ there is an open ball in $\mathbb{R}^n$ which contains $x$ but no other point of $X$. 

(1) Show that if the orders of two elements $x$ and $y$ in $G$ are relatively prime, then the order of $xy$ is the product of the orders of $x$ and $y$.

(2) Let $m$ be the largest number which is the order of some element in $G$. Show that the order of every element of $G$ divides $m$.

7. Finite groups of rigid motions of $\mathbb{R}^3$. Let $G$ be a finite group of orientation preserving isometries of three-space.

(1) Explain why $G$ is isomorphic to a subgroup of $SO_3(\mathbb{R})$, and therefore every non-trivial element of $G$ can be identified with rotation around some line $\ell$ through the origin. [This is just stringing together other things we have done, not a long argument from scratch.]

(2) Let $L$ be the set of all lines in $\mathbb{R}^3$ that are the axis of rotation for some non-trivial element $g \in G$. Show that the tautological action of $G$ on $\mathbb{R}^3$ naturally induces an action on $G$ on $L$. [Hint: First show that if $g$ and $g_1$ are invertible linear transformations of some vector space, and $v$ is an eigenvector for $g_1$, then $g(v)$ is an eigenvector for linear transformation $g \circ g_1 \circ g^{-1}$.

(3) Show that there is a naturally induced action on the set $P$ (called ”poles”) consisting of points on the unit sphere which lie on some $\ell \in L$.

[This is the beginning of the proof of the classification of all finite groups of spacial isometries. It turns out there are only five up to isomorphism. The rest is interesting too, see Artin Chapter 5 Section 9, but I thought maybe you had a long enough assignment already.]

Problems from Artin: 5.4 #14; 5.5 #6, #8; 5.6 #5; 5.8 #1, 3, 5