CHOOSE ONE:

1. (2.2 # 16 a from Artin): Let $G$ be a cyclic group of order 6. How many of its elements generate $G$?

OR

2. Prove that every subgroup of $\mathbb{Z}$ is cyclic.

Answers.

1. Say $x$ generates $G$. Then the elements of $G$ are $\{x, x^2, x^3, x^4, x^5, x^6 = e\}$. We claim that only $x$ and $x^5$ generate $G$. Indeed, $x$ obviously does, and since $x^5 = x^{-1}$, $x$ and $x^5$ necessarily generate the same subgroup. Using the fact that $x^6 = e$, we check that the subgroup generated by $x^2$ contains the three elements $x^2$, $(x^2)^2 = x^4$, $e$, so that $x^2$ is not a generator. Since the inverse of $x^2$ is $x^4$, neither does $x^4$ generate $G$. Likewise, $x^3$ generates a two element subgroup containing just $x^3$ and $e$. Thus the only two elements of $G$ that generate it are $x$ and $x^5$.

2. It suffices to show that if $G$ is a subgroup of $\mathbb{Z}$, then $G$ is generated by some $n \in \mathbb{Z}$. If $G$ is the trivial subgroup, then it is generated by 0. Assume $G$ is not (0). Then it contains some non-zero element $x$, and also its inverse $-x$, so that $G$ must contain a positive integer. By the well-ordering principle, the non-empty subset $\mathbb{N} \cap G$ has a minimal element, call it $n$. We claim that $n$ generates $G$. To see this, take any $x \in G$. By the division algorithm, we can write $x = qn + r$ for some unique integers $q, r \in \mathbb{Z}$ satisfying $0 \leq r < n$. Now, since $G$ is closed under addition and inverses, and both $n$ and $x$ are in $G$, we have also that $x - qn = x - (n + \cdots + n)$ is in $G$. Thus $r$ is in $G$. This contradicts the minimality of $n$, unless $r = 0$. We conclude that $x = qn$, which is to say, $G = n\mathbb{Z}$ is generated by $n$. 