1. Show that if $R$ is Noetherian, then for any ideal $J \subset R$, the ring $R/J$ is also Noetherian.

2. **Hilbert’s Basis Theorem.** Let $R$ be a Noetherian ring. Hilbert’s Basis Theorem says that $R[X]$ is also Noetherian.¹
   (1) Using Hilbert’s basis theorem, deduce that $\mathbb{F}[X_1, \ldots, X_n]$ is Noetherian, where $\mathbb{F}$ is a field.
   (2) Prove that every affine algebraic set (as defined on Homework Set 2) in $\mathbb{P}^n$ is the common zero set of finitely many polynomials.
   (3) Prove that every affine algebraic set in $\mathbb{P}^n$ is the intersection of finitely many hypersurfaces, where a hypersurface, by definition, is the zero set $V(f)$ of a single polynomial $f$ in $n$ variables.

3. **The coordinate ring of an affine algebraic variety.** Let $V \subset \mathbb{C}^n$ be any affine algebraic set. The coordinate ring of $V$ is the ring $\mathbb{C}[V]$ of all (complex valued) functions on $V$ which are the restriction to $V$ of some complex polynomial function on $\mathbb{C}^n$.
   (1) Show that $\mathbb{C}[V]$ really is a ring, with an appropriate interpretation of $+$ and $\times$.
   (2) Find a natural surjective ring homomorphism $\mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[V]$, and prove that its kernel is the ideal $I_V$ of all polynomials that vanish on $V$.
   (3) Prove that $\mathbb{C}[V]$ is Noetherian.

4. **Cyclic Modules.** An $R$-module is cyclic if it can be generated by one element.
   (1) Show that for any non-zero ideal $I$ in any domain $R$, the module $R/I$ is always cyclic but never free.
   (2) Find and prove a criterion, in terms of $n$ and $m$, such that the $\mathbb{Z}$-module $\mathbb{Z}_n \oplus \mathbb{Z}_m$ is cyclic.
   (3) Show that every cyclic $R$-module is isomorphic to a module of the form $R/I$ where $I$ is an ideal of $R$.

5. **Modules over non-commutative rings.** Let $R$ be a non-commutative ring.
   (1) Prove or disprove: if $I$ is a left ideal in $R$, then $R/I$ is a (possibly non-commutative) ring.
   (2) Prove or disprove: if $I$ is a left ideal in $R$, then $R/I$ is an $R$-module. [Caution!]
   (3) Our definition of $R$-module is better called a left $R$-module in the case where $R$ is not necessarily commutative. Define right $R$-module. [Caution!]
   (4) Prove or disprove: if $I$ is a right ideal in a non-commutative ring $R$, then $R/I$ is an left $R$-module. Is $R/I$ a right $R$-module?
   (5) Prove that the kernel of a homomorphism of left $R$-modules is a left $R$-module. Is analogous true for right modules and right ideals?
   (6) Prove a first isomorphism theorem for left $R$-modules. Is there a first isomorphism theorem for right $R$-modules?

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¹This may eventually be assigned for you to prove. I took it off the assignment for this week in order to keep the assignment a reasonable length.

(1) Consider the set \( \text{End}_\mathbb{R}(\mathbb{R}[x]) \) of all \( \mathbb{R} \)-vector space homomorphisms \( \mathbb{R}[x] \rightarrow \mathbb{R}[x] \). Show that it has a natural non-commutative ring structure.

(2) Show that the map \( \mathbb{R}[x] \rightarrow \text{End}_\mathbb{R}(\mathbb{R}[x]) \) sending \( f \) to the map “multiplication by \( f \)” is an injective homomorphism of (nnc) rings. Note that this lets us view \( \mathbb{R}[x] \) as a subring of \( \text{End}_\mathbb{R}(\mathbb{R}[x]) \).

(3) Let \( D = \mathbb{R}\langle x, \frac{\partial}{\partial x} \rangle \) be the subring of \( \text{End}_\mathbb{R}(\mathbb{R}[x]) \) generated by \( \mathbb{R}[x] \) and \( \frac{\partial}{\partial x} \). Show that we have the relation \( \frac{\partial}{\partial x} \circ x - x \circ \frac{\partial}{\partial x} = 1 \) in \( D \). This ring \( D \) is called the Weyl algebra.

(4) Explain why the Weyl algebra really is an algebra (over the field \( \mathbb{R} \)).

(5) Explain how we can view \( \mathbb{R}[x] \) in a natural way as a (left) \( D \)-module. Show that it is cyclic, but is not free.

(6) Find a natural surjective homomorphism \( D \rightarrow \mathbb{R}[x] \) of left \( D \)-modules and compute its kernel. What kind of mathematical object is this kernel? What does the first isomorphism theorem say in this case?

Artin: Chapter 12, §1: 6, 8, 9; §2: 1, 4, 5, 7; §4: 3