1. Cokernels of diagonal matrices. Let $R$ be any ring, and let $A$ be an $m \times n$ matrix which is diagonal with the (non-zero) elements $d_1, d_2, \ldots, d_t$ on the diagonal and zeros elsewhere. Consider the map given by (left) multiplication (of column vectors) by $A$:

$$\phi_A : R^n \to R^m.$$ 

Prove that the cokernel of $\phi_A$ is isomorphic to $R/(d_1) \oplus \cdots \oplus R/(d_t) \oplus R^{m-t}$. Stated differently: prove that the matrix $A$ presents the module $R/(d_1) \oplus \cdots \oplus R/(d_t) \oplus R^{m-t}$. [Hint: don’t forget or reinvent the isomorphism theorems!]

2. Comparing the Zariski topology and the Euclidean topology on $\mathbb{C}^n$.

1. Show that every affine algebraic set of $\mathbb{C}^n$ is closed in the Euclidean topology of $\mathbb{C}^n$. [Hint: Use the fact that a polynomial over $\mathbb{C}$ is continuous in the Euclidean topology.]

2. Given two topologies (call them $T$ and $T'$) on a set $X$, we say that $T$ is **coarser** than $T'$ if every $T'$-open set is open in the topology $T$. Show that the Zariski topology is coarser than the Euclidean topology.

3. A topological space $X$ is Hausdorff if any pair of distinct points, $x_1$ and $x_2$, there exist disjoint open sets $U_1$ and $U_2$ such that $x_i \in U_i$. Is the Zariski topology on $\mathbb{C}^n$ Hausdorff? Is the Euclidean topology Hausdorff?

3. Affine Schemes. Let $R$ be a ring, and let $\text{Spec } R$ be the collection of all prime ideals of $R$. For each ideal $I \subset R$, let $\mathcal{V}(I)$ be the subset of $\text{Spec } R$ consisting of all prime ideals containing the ideal $I$.

1. Prove that if $P$ is a prime ideal in a ring $R$, and $IJ \subset P$ for some ideals $I$ and $J$ of $R$, then $I \subset P$ or $J \subset P$. Interpret this as a familiar (to high school students) statement when $R = \mathbb{Z}$.

2. Prove that $\text{Spec } R$ can be given the structure of a topological space whose closed sets are the subsets of the form $\mathcal{V}(I)$. This is the Zariski topology on $\text{Spec } R$. Such a topological space is called an **affine scheme**.

3. Explicitly describe the Zariski topology on $\text{Spec } \mathbb{Z}$ and on $\text{Spec } \mathbb{C}[X]$.

4. Show that $\text{Spec } \mathbb{Z}$ and on $\text{Spec } \mathbb{C}[X]$ contain a dense point, that is, a point whose closure is the whole space. Is the same true in $\mathbb{R}$ with the standard Euclidean topology?

5. Is $\text{Spec } \mathbb{Z}$ Hausdorff?