
1. GROUP ACTIONS

Let \((G, \circ)\) be any group. Let \(X\) be any set (finite or infinite). An action of \(G\) on \(X\) is a natural way that the group elements “move around” the elements of \(X\). Formally:

**Definition 1.1.** An action of \(G\) on \(X\) is a map \(G \times X \to X\) \((g, x) \mapsto g \cdot x\)

which satisfies

1. \(e \cdot x = x\) for all \(x \in X\).
2. \(g_1 \cdot (g_2 \cdot x) = (g_1 \circ g_2) \cdot x\) for all \(g_1, g_2 \in G\) and all \(x \in X\).

Here, it is important to absorb the correct use of notation: the set \(X\) has no extra structure (no operation) but the action allows us to combine a group element \(g \in G\) with a set element \(x \in X\) to get a new set element \(g \cdot x \in X\).

The two axioms of an action ensure that this procedure is compatible with the group structure. In particular, the first axiom tells us that the identity element of \(G\) behaves as expected: it does nothing to any \(x \in X\).

Look carefully at Axiom 2: there are two different notations, - and \(\circ\), and they mean two different things. Axiom 2 tells us that if we move an element \(x \in X\) to another element of \(X\) using first \(g_2\) and then \(g_1\), the result is the same as if we let \(g_1 \circ g_2\) act directly on \(x\).

**Example 1.2.** Let \(G\) be the group \(GL_2(\mathbb{R})\) of \(2 \times 2\) matrices with real coefficients. Let \(X = \mathbb{R}^2\) be the set of column vectors \(\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}\).

Then \(G\) acts on the set \(X\) in a natural way by left multiplication:

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.
\]

You should check that both axioms of a group action hold: the first holds because of the properties of the identity matrix in \(GL_2(\mathbb{R})\) and the second because of the associative law for matrix multiplication.

**Example 1.3.** Consider the group \(G = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}\) under addition. This group acts on the Cartesian plane \(\mathbb{R}^2\) in an obvious way:

\((m, n) \cdot (x, y) = (m + x, n + y)\)

by translation. It is easy check that this is an action. (Do it!)

2. ACTIONS ON OBJECTS IN MORE INTERESTING CATEGORIES.

Often, we have a group action on a set \(X\) where \(X\) has some additional structure—that is, the set \(X\) may be a topological space, a vector space, an abelian group, or an object in some other category. In this case, we may want the group action to preserve this additional structure, that is, we may want \(G\) to act on \(X\) by continuous maps if \(X\) is a topological space, or by linear transformations in \(X\) is a vector space, and so on. For example, the set \(\mathbb{R}^2\) can be considered as a real vector space, as a topological space, as an abelian group (under \(+\)), as a smooth manifold, or as a metric space. A group action on the underlying set \(\mathbb{R}^2\) may or may not respect one or more of these additional structures.
Consider Example 1.2, with the group \(G = GL_2(\mathbb{R})\) acting on the set \(\mathbb{R}^2\) by matrix multiplication. That is, each \(g \in G\) acts on \(\mathbb{R}^2\) by the linear transformation \(
abla \begin{bmatrix} x \\ y \end{bmatrix} \mapsto g \begin{bmatrix} x \\ y \end{bmatrix} \). This action of \(G\) on \(\mathbb{R}^2\) respects the vector space structure of \(\mathbb{R}^2\). That is, this is an action of \(G\) on \(\mathbb{R}^2\) as a vector space or in the category of vector spaces.

In Example 1.3, however, the group \(G = \mathbb{Z}^2\) is acting on \(\mathbb{R}^2\) by translation: an element \(g = (m, n)\) gives a map \(
abla \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} m + x \\ n + y \end{bmatrix} \), which is not a linear transformation of \(\mathbb{R}^2\). So this action is not respecting the vector space structure of \(\mathbb{R}^2\). This action is not an action in the category of real vector spaces.

On the other hand, we can view \(\mathbb{R}^2\) as a topological space instead. Since linear transformations are continuous maps, the action of \(GL_2(\mathbb{R}^2)\) on \(\mathbb{R}^2\) described in Example 1.2 preserves the structure of the topological space \(\mathbb{R}^2\). Likewise, since translations in \(\mathbb{R}^2\) are continuous maps, the action of \(\mathbb{Z}^2\) on \(\mathbb{R}^2\) described in Example 1.3 is an action on \(\mathbb{R}^2\) as a topological space as well.

You should check for yourself that the action in Example 1.2 can be viewed as an action in the category of smooth manifolds, as well as an action in the category of abelian groups, but not, for example, in the category of metric spaces (if we give \(\mathbb{R}^2\) its standard metric structure). The action in Example 1.3 is an action in the category of smooth manifolds but not in the category of abelian groups. Is it an action on \(\mathbb{R}^2\) as a metric space?

### 3. Another Point of View on Group Actions

Perhaps the most fundamental example of a group is the group of bijections of a fixed set \(X\).

Fix any set \(X\). Consider the set of all bijections from \(X\) to itself. The composition of two bijections \(X \xrightarrow{g_1} X\) and \(X \xrightarrow{g_2} X\) is certainly another bijection \(X \xrightarrow{g_2 \circ g_1} X\). By definition, every bijection \(X \xrightarrow{g} X\) has an inverse \(X \xrightarrow{g^{-1}} X\), and of course, the identity map \(X \xrightarrow{\epsilon} X\) is a bijection. So you can easily check that the set of all bijections from a set \(X\) to itself forms a group under composition.

If \(X\) is finite, the group of all bijections from \(X\) to itself is usually called the **permutation group** of \(X\), which is perhaps not an unreasonable name in general. Formally, the group of all bijections from \(X\) to itself is the **automorphism group of \(X\) in the category of Sets.** So we will denote this group \(\text{Aut}_{\text{Set}}(X)\).

Now suppose that a group \(G\) acts on a set \(X\). One way to understand the action is to imagine picking one \(g \in G\) and thinking about where that \(g\) sends each element of \(X\). Each element of \(G\) gives rise to a mapping

\[
X \xrightarrow{\phi_g} X \quad x \mapsto g \cdot x.
\]

This mapping is a bijection, because it has inverse

\[
X \xrightarrow{\phi^{-1}_g} X \quad x \mapsto g^{-1} \cdot x.
\]

So each element of \(G\) determines a **bijection** of \(X\) to itself. Putting this together, we are saying that if a group \(G\) acts on a set \(X\), we have a natural mapping

\[
G \rightarrow \text{Aut}_{\text{Set}}(X) \quad g \mapsto \phi_g.
\]

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1The category of Sets has as its morphisms set mappings; that is, given two sets \(X, Y\) (objects in the category **Set**), a morphism between them is simply a mapping \(X \rightarrow Y\). An isomorphism in the category of sets is therefore a bijection between \(X\) and \(Y\). An automorphism of an object \(X\) in the category **Set** is therefore a bijection from \(X\) to itself.
This map is sometimes called the **adjunction map** determined by the group action. The axioms for a group action (of $G$ on $X$) translate into the fact that the adjunction mapping is a **group homomorphism**. It is critical to your understanding that you verify this!

For example, the action in Example 1.2 gives rise to the group homomorphism

$$GL_2(\mathbb{R}) \to \{\text{Bijections from } \mathbb{R}^2 \text{ to } \mathbb{R}^2\}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \text{the map } \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$  

Conversely, any group homomorphism $\Phi : G \to \text{Aut}_{\text{Set}}(X)$ gives us a natural way to define an action on $G$ on the set $X$. Take a moment to make sure you see how. Given the group homomorphism $\Phi$, to what element of $X$ can you send the pair $(g, x)$? Why does this assignment satisfy the axioms of a group action?

In fact, an action of $G$ on $X$ is essentially the same exact thing as a group homomorphism $G \to \text{Aut}_{\text{Set}}(X)$. You should unravel why and how this is so. Our textbook goes back and forth between these points of view without comment, so you should internalize this idea. (See Theorem 3.1 below for details, if needed. But try it yourself first, as I think this kind of thing is easier to do yourself than to read.)

The adjunction mapping clarifies what it means that $G$ acts on a topological space $X$. We can view each homeomorphism $X \overset{g}{\to} X$ as a special kind of bijection from $X$ to itself. Since the composition of homeomorphisms is a homeomorphism, the self-homeomorphisms of a topological space form a group $\text{Homeo}(X)$, which is of course the automorphism group of $X$ in the category of topological spaces, so we can also denote it $\text{Aut}_{\text{Top}}(X)$. The statement that a homeomorphism between topological spaces is a bijection with certain extra properties is basically a natural inclusion of groups

$$\text{Homeo}(X) = \text{Aut}_{\text{Top}}(X) \subset \text{Aut}_{\text{Set}}(X).$$

To say that $G$ acts on $X$ by homeomorphisms (or as a topological space), is simply to say that the adjunction map described above takes values in the subgroup $\text{Aut}_{\text{Top}}(X)$. Likewise, if $X$ is a real vector space, then to say that $G$ acts on the real vector space $X$ is to say that adjunction mapping lands in the subgroup $\text{Aut}_{\text{Vec}}(X)$ of $\text{Aut}_{\text{Set}}(X)$.

### 3.1. Formal details.

**Theorem 3.1.** Let $G$ be a group acting a set $X$. Then there is an induced group homomorphism

$$G \overset{\Phi}{\to} \text{Aut}_{\text{Set}}(X) \quad g \mapsto [X \overset{g}{\to} X \quad x \mapsto g \cdot x],$$

where $\text{Aut}_{\text{Set}}(X)$ denotes the group of all bijections from $X$ to itself, under composition. Conversely, given a group homomorphism

$$G \overset{\Phi}{\to} \text{Aut}_{\text{Set}}(X)$$

we can recover a group action of $G$ on $X$ as follows: $g \cdot x = \Phi(g)(x)$.

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But more often, special notation for it is suppressed and the reader is expected to go seemlessly between these points of view.
Proof. Fix an action of a group $G$ on a set $X$. We need to prove that the adjunction map is a group homomorphism. That is, we must show that for any $g, h \in G$, $\Phi(gh) = \Phi(g) \circ \Phi(h)$. By definition of $\Phi$, this says we must show that $\phi_{gh} = \phi_g \circ \phi_h$. These are two different bijections $X \to X$, so to show that they are equal, we must show that for every input $x \in X$, they have the same output.

By definition, $\phi_{gh}(x) = (gh) \cdot x$. On the other hand $\phi_g \circ \phi_h(x) = g \cdot (h \cdot x)$. So we must verify that $(gh) \cdot x = g \cdot (h \cdot x)$. But this is one of the axioms of a group action! Thus $\Phi$ is a group homomorphism.

For the other direction, assume that $\Phi$ is a group homomorphism. Then $\Phi(e)$ is the identity in $\text{Bij}(X)$. This means that $\phi_e$ does nothing to any $x$, or in other words $e \cdot x = x$ for all $x \in X$. This verifies the first axiom of a group action.

For the second axiom, we use that $\Phi(gh) = \Phi(g) \circ \Phi(h)$. This means that $\phi_{gh} = \phi_g \circ \phi_h$, which again means that $gh \cdot x = g \cdot (h \cdot x)$ for all $x \in X$. The second axiom is verified. QED.

Example 3.2. As an application of the adjunction mapping, we can better understand the rotational symmetry group $G$ of the cube. Note that $G$ acts naturally on the set of 4 grand diagonals of the cube. The action of $G$ on this four-element set is equivalent to a group homomorphism $G \to S_4$.

Since both groups $G$ and $S_4$ have 24 elements, this map will be an isomorphism if it is injective (or surjective). To check that it is injective, consider an element $g \in G$ is in the kernel. This means $g$ induces trivial bijection— the identity map—on the set of grand diagonals. In other words, $g$ is in the stabilizer of each grand diagonal. But the stabilizer of each grand diagonal is an order three group of rotations around that diagonal. The intersection of any two of these stabilizers is $\{e_G\}$ (since the intersection is a subgroup of both), so the entire kernel is trivial. This proves that $G \cong S_4$. The symmetry group of a cube is isomorphic to $S_4$!