Math 593 Exam II Fall 2005

1. (18 points) Prove or disprove the following three statements, where R is a commutative ring.

a). Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. If A and C are finitely generated, then B is finitely generated.

TRUE. Let $a_1, \ldots, a_n$ be generators for A and $c_1, \ldots, c_m$ be any lifts to B of generators $\bar{c}_1, \ldots, \bar{c}_m$ of C via the surjective map $B \to C$. Then B is generated by $\{a_1, \ldots, a_n, c_1, \ldots, c_m\}$. Indeed, take an arbitrary element $b \in B$. Its image in C can be written $\sum_{i=1}^m r_i \bar{c}_i$. Then the element $b - \sum_{i=1}^m r_i c_i$ of B is in the kernel of $B \to C$, which means it is some R-linear combination of the elements $a_1, \ldots, a_n$. So an arbitrary element of B is in the R-span of $\{a_1, \ldots, a_n, c_1, \ldots, c_m\}$. QED.

b). If M is a non-cyclic R-module, then $\bigwedge^2 M$ is non-zero.

FALSE. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. It is not cyclic yet $\mathbb{Q} \wedge \mathbb{Q}$ is zero.

c). If M is any module over a domain R, then the R-module $\mathrm{Hom}_R(M, R)$ is torsion free.

TRUE. Say $\phi : M \to R$ is killed by some non-zero r. Then for all $m \in M$, we have $r\phi(m) = 0$. But R is a domain, so this forces $\phi(m) = 0$ for all m, which means ϕ is the zero map. QED

2. (16 points) Find all prime ideals in the ring

$$\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \left(\mathbb{Z}[x,y]/(x^3) \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y]/y^2\right).$$

We have isomorphisms

$$\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \left(\mathbb{Z}[x,y]/(x^3) \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y]/y^2\right) \to \mathbb{Z}/10[x,y]/(x^3, y^2) \to \mathbb{Z}[x,y]/(10, x^3, y^2)$$

The first arrow comes from a general fact proven in class, the second uses the exact same idea (or you can apply a general "extension of scalars" fact proved in the homework). Any prime in this ring corresponds to a prime in $\mathbb{Z}[x,y]$ containing $(10, x^3, y^2)$. By definition of primeness, since such a prime must contain also x and y, and either 2 or 5. So any prime must contain either $(x, y, 2)$ or $(x, y, 5)$. But both these ideals are maximal (as taking their quotients produces the fields $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$ respectively). Hence these only two prime ideals of our ring are the images of these two in the quotient ring $\mathbb{Z}[x,y]/(10, x^3, y^2)$.

[If you want, you can write them as $(\bar{2} \otimes \bar{1} \otimes \bar{1}, \bar{1} \otimes \bar{x} \otimes \bar{1}, \bar{1} \otimes \bar{1} \otimes \bar{y})$ and $(\bar{5} \otimes \bar{1} \otimes \bar{1}, \bar{1} \otimes \bar{x} \otimes \bar{1}, \bar{1} \otimes \bar{1} \otimes \bar{y})$, but this is not really necessary.]

3. (16 points) Let M and L be modules over a commutative ring R, and let $\mathrm{SYM}^k(M,L)$ be the set of k-multilinear symmetric maps over R

$$M \times M \times \ldots \times M \to L.$$

a). Describe a natural R-module structure on the set $\mathrm{SYM}^k(M,L)$.

We add ϕ and ψ by adding their values in L, and also multiply by elements of R by multiplying the values in L by elements of R. These maps are still obviously symmetric, and the R-module structure on L obviously induces a R-module structure on the set $SYM(S^kM, L)$.

b). Show that there is a R-module isomorphism $\mathrm{SYM}^k(M,L) \cong \mathrm{Hom_R}(S^kM, L)$.

Given an element of ϕ of $\mathrm{SYM}^k(M,L)$, we use the universal property of $S^k(M)$ to construct a unique R-module map $f(\phi)$ from $S^k(M)$ to L. We claim that the asssociation f is an R-module map: indeed, since $f(\phi+\psi)$ sends the class of the tensor $m_1 \otimes \ldots \otimes m_k$ to $\phi(m_1,\ldots,m_k) + \psi(m_1,\ldots,m_k)$, whereas $f(\phi)$ (respectively $f(\psi)$) sends the class of the tensor $m_1 \otimes \ldots \otimes m_k$ to $\phi(m_1,\ldots,m_k)$ (respectively $f(\psi)$), and such classes of tensors generate $S^k(M)$, we clearly have $f(\phi+\psi) = f(\phi) + f(\psi)$. Likewise, $f(r\phi)$ sends the class of the tensor $m_1 \otimes \ldots \otimes m_k$ to $r\phi(m_1,\ldots,m_k)$, which is clearly the same $rf(\phi)$ applied to the same class. Thus f is an R-module homomorphism. Finally, f is injective, since if $f(\phi) = 0$, then from the commutative diagram in the universal property it follows that ϕ, which is the composition $M \times \ldots \times M \to S^k(M)\overline{f(\phi)} \to L$ is also zero. It is also surjective: given any $\Phi \in \mathrm{Hom}(S^kM, L)$, the composition $M \times \ldots \times M \to S^k(M)\overline{\Phi} \to L$ is an element ϕ of $\mathrm{SYM^k}(M,L)$, so by uniqueness of the map $f(\phi)$, we have $f(\phi) = \Phi$.

4. (16 points) Let A, B, and C be finite dimensional vector spaces over a field F of dimensions a, b and c, respectively. Compute the dimension of

$$\left[\mathrm{Hom}_F(\bigwedge^k A,\ S^\ell B) \otimes_F (T^m C)^*\right] \bigoplus \bigwedge^p (A \bigoplus B).$$

Since $\bigwedge^k A$ has dimension $\binom{a}{k}$, and $S^\ell B$ has dimension $\binom{b+\ell-1}{b-1}$, clearly $\mathrm{Hom}_F(\bigwedge^k A,\ S^\ell B)$ has dimension $\binom{a}{k}\binom{b+\ell-1}{b-1}$. Since (T^mC) has dimension c^m, so does its dual. Also, $\bigwedge^p(A \bigoplus B)$ has dimension $\binom{a+b}{p}$. Putting this all together, we get the dimension is

$$\binom{a}{k}\binom{b+\ell-1}{b-1}c^m + \binom{a+b}{p}.$$

5. (16 points) Let V and W be finite dimensional vector spaces over a field F of dimensions m and n respectively. Fixing bases $\{v_1, v_2, \ldots, v_m\}$ for V and $\{w_1, v_2, \ldots, w_m\}$ for W, consider the map

$$V \times W \to M_{m\times n}(F)$$

$$(v, w) \mapsto \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix}$$

where $v = a_1v_1 + a_2v_2 + \ldots + a_mv_m$ and $w = b_1w_1 + b_2w_2 + \ldots + b_nw_n$.

a). Show that this map induces an isomorphism of F-vector spaces, $V \otimes W \cong M_{m\times n}(F)$.

The map is bilinear over F because matrix multiplication is bilinear. Thus it induces a F-vector space map $T : V \otimes W \to M_{m\times n}(F)$. Note that this linear transformation sends the element $v_i \otimes w_j$ to the matrix whose ij-entry is 1 and all other entries are zero. Thus T sends a basis of $V \otimes W$ to a basis of $M_{m\times n}(F)$, so must be an isomorphism.

b). Prove that under this isomorphism, the simple tensors $v \otimes w$ are in one-to-one correspondence with the rank one $m \times n$ matrices.

A simple tensor $v \otimes w$ is sent to the $n \times n$ matrix $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix}$, whose rows are clearly all scalar multiples of the row matrix $\begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix}$. Thus the rows are all linearly dependent, and the matrix is rank one. Conversely, given a rank one matrix, its rows are all scalar multiples of eachother. Some row, which without loss of generality we can assume to be the first, is non-zero. So labling the rows $R_1, R_2, \ldots, R_m$, there exist scalars $a_2 dots, a_m$ such that $a_iR_i = R_1$ for all $i \geq 2$. If R_1 is the row matrix $(b_1, \ldots, b_n)$, this exactly means that the matrix factors as $\begin{pmatrix} 1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix}$, and so corresponds to the simple tensor $\sum_{i=1}^m a_iv_i \otimes \sum_{j=1}^n b_jw_j$ under the transformation T (where $a_1 = 1$).

c). Assuming $n, m \geq 2$, find an explicit element in $V \otimes W$ that is not simple.

The tensor $v_1 \otimes w_1 + v_2 \otimes w_2$ is not simple, from parts a and b.

[Note that this exercise shows that, at least if m and n are greater than one and $F = \mathbb{R}$, the simple tensors form a "set of measure zero" among all tensors.]

6. (16 points) Let R be a commutative ring, and I and ideal in R. Let S be the subset of the polynomial ring $R[t]$ consisting of polynomials $f(t) = \sum a_i t^i$ where $a_i \in I^i$.

a). Prove that there is a surjective R-algebra map from the symmetric algebra $\mathcal{S}(I)$ of the R-module I onto S.

We note first that S is a commutative R-algebra. Now define a map $I \to S$ sending $x \mapsto xt$. Since $(rx)t = r(xt)$ and $(x+y)t = xt + yt$, this is clearly an R-module map. By the universal property of the symmetric algebra, it extends to an R-algebra map $\Phi : \mathcal{S}(I) \to S$. It remains only to show that this map is surjective. For this, note that I^k is generated by products of the form $x_1 \dots x_k$, where each $x_i \in I$. Thus S is generated (as an R-module) by polynomials of the form $x_1 \dots x_k t^k$, as we range over all values of $k \geq 0$. Such an element is the image, under Φ, of the class of the simple tensor $x_1 \otimes \dots \otimes x_k$ in $S^k(M) \subset \mathcal{S}(M)$.

b). In the case where R is a PID, prove that this map is an isomorphism.

If I is the zero ideal, the result is trivial, as both rings are just R. Otherwise, $I = (a)$ for some $a \in R$, and hence is a free R-module of rank 1. Hence each $S^k(I)$ is a free R-module of rank 1 as well, with generator the class of $a \otimes a \otimes \dots \otimes a$ (k times), let us denote this generator by $\otimes^k a$. So an arbitrary element of $S(I)$ is of the form $\sum_{i=0}^k r_i(\otimes^i a)$, and under Φ this element maps to $\sum_{i=0}^k r_i a^i t^i$. So if such an element were sent to zero, each coefficient $r_i a^i$ must be zero. But a^i is a non-zero element in a domain R, so this forces each $r_i = 0$. This shows the kernel of Φ is trivial, concluding the proof that Φ is an isomorphism.

EXTRA CREDIT: Give an example where this map is not an isomorphism.

I will still accept solutions for this.

[The ring S is called the Rees ring of the ideal I, and plays an important role in the operation of "blowing up" in algebraic geometry.]