MATH 593: Tenth Homework Assignment: More on Rational and Jordan Canonical Form

Due Wednesday November 23, 2005

This week's reading:

Please read the handout from Rotman, Advanced Modern Algebra, by Monday November 21.

1*. Let T be the transformation of $\mathbb{C}^3$ given by the matrix

$$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{pmatrix}$$

acting by left multiplication on columns. Let $\bigwedge^2 T$ be the induced transformation of $\bigwedge^2 \mathbb{C}^3$.

a). What are the possible Jordan Canonical forms of $\bigwedge^2 T$ and for what values of a and b does each occur?

b). Find necessary and sufficient conditions such that $\bigwedge^2 T$ is diagonalizable.

c). Find necessary and sufficient conditions such that $\bigwedge^2 T$ is nilpotent.

2*. Let A be a matrix with entries in a Euclidean domain. Define $d_i(A)$ to be the greatest common divisor of the $(i \times i)$ minors of A. Set $\sigma_i(A) = d_i(A)/d_{i-1}(A)$ (where $d_0 = 1$.)

a). Prove that the Smith Normal form of A has the elements σ_i as its ii entry and zeros elsewhere.

b). Use this result to find the rational canonical form of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

3*. A lie algebra over a field F is a vector space L equipped with an alternating bilinear map $L \times L \rightarrow L$, called the "bracket" and denoted $(a, b) \mapsto [a, b]$ satisfying the *Jacobi identity:*

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all a, b, c in L.

a). Verify that $M_{n \times n}(F)$ is a lie algebra where the bracket is defined as $[a, b] = ab - ba$. This "quintessential" lie algebra is usually denoted $\mathfrak{gl}(n, F)$. (It is a fact that every finite dimensional lie algebra is a sub-lie algebra of this one, if F is a field of characteristic zero).

b). Prove that every element of the lie algebra $\mathfrak{gl}(n, \mathbb{C})$ can be written as $d + n$ where d is a diagonalizable matrix, n is a nilpotent matrix, and $[d, n] = 0$. (This fact is very useful in the study of lie algebras.)

4*. a.) Let R be a PID, a, b non-zero elements of R, whose greatest common divisor is d. Prove that there is a 2×2 matrix Q with determinant 1 such that

$$Q \begin{pmatrix} a & * \\ b & * \end{pmatrix} = \begin{pmatrix} d & * \\ d' & * \end{pmatrix}$$

where d divides d'.

b). Call a matrix secondary if it can be put into block diagonal form where one of the blocks is a 2×2 matrix of determinant one and all other blocks are 1×1 identity matrices. Prove that every $n \times n$ matrix A can be transformed into Smith normal form by a sequence of multiplications by secondary and elementary matrices on the left and right.

c). Use b) to give a proof of the existence part of "The Theorem" for an arbitrary PID along the lines we did in class for Euclidean domains. That is: Use b) to show that if M is a finitely generated module over a PID R, then M is isomorphic to $R^r \oplus R/(a_1) \oplus \ldots R/(a_m)$ where $(a_1) \subset (a_2) \subset \ldots (a_m)$.

Then from Dummit and Foote:

Section 12.3 : 1, 2, 4, 5, 6, 7, 8, 9, 10, 11*, 12, 13, 14, 15, 17*, 18, 19, 20, 21, 22, 23, 24, 27, 29, 31, 33, 34, 35, 36, 37, 38*,

NOTE: The final exam is scheduled (by the registrar's office) for Wednesday Dec 21 at 4 pm. Place TBA.