MATH 593: Eleventh Homework Assignment: Bilinear Forms

Due Monday December 5, 2005

This week's reading:

Please read the handout from Rotman, Advanced Modern Algebra, by Monday November 21.

1*. Consider linear transformations $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$. Note that there is a naturally induced linear transformation $T_1 \otimes T_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$.

a). If A_1 denotes the matrix of T_1 with respect to the bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ for V_1 and W_1 respectively, and A_2 denotes the matrix of T_2 with respect to the bases $\{v'_1, \ldots, v'_p\}$ and $\{w'_1, \ldots, w'_q\}$ for V_2 and W_2 respectively, describe the matrix for $T_1 \otimes T_2$ in terms of the bases $\{v_i \otimes v'_j\}$ and $\{w_k \otimes w'_l\}$ for $V_1 \otimes V_2$ and $W_1 \otimes W_2$ respectively. This is called the tensor product of the matrices A_1 and A_2 and denoted $A_1 \otimes A_2$.

b). Prove that the tensor product of two positive definite square symmetric matrices over $\mathbb{R}$ is positive definite.

c). Let Q_1 and Q_2 be quadratic forms on finite dimensional real vector spaces V_1 and V_2 respectively. Show that there is a quadratic form on $V_1 \otimes V_2$ satisfying $Q(v_1 \otimes v_2) = Q_1(v_1)Q_2(v_2)$.

d). Find the rank and signature of Q from part c in terms of those for Q_1 and Q_2.

2.* After an orthonormal change of basis (over $\mathbb{R}$), the quadratic form $2x_1x_2 + 2x_2x_3 + 6x_3x_4$ becomes $a_1x^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ for some a_i.

a.) Compute explicitly the values $a_1 + a_2 + a_3 + a_4$ and $a_1a_2a_3a_4$ without computing the a_i.

b). Is this quadratic form positive definite, negative definite, and/or non-degenerate?

3.* A real symmetric 4×4 matrix has characteristic polynomial $x^4 - dx + 12$ for some real number d. Is the corresponding quadratic form postive definite, negative definite, and/or non-degenerate? How many eigenvalues are positive?

4.* Let ϕ be an alternating bilinear form on an n-dimensional vector space V. Assume that the maximal dimension of a subspace $W \subset V$ such that $\phi(x, y) = 0$ for all x, y in W is zero is m (such a subspace is called *isotropic*.) Find the rank of ϕ.

5.* Let A be a real matrix of any dimensions. Prove that $A^{tr}A$ and AA^{tr} are positive semi-definite and have the same non-zero eigenvalues with the same multiplicities.

6.* Consider the real matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & a \end{pmatrix}$.

a). For each a, determine the rank and signature of a bilinear form with matrix A.

b). For which values of a is A diagonalizable?

c). For which values of a is A positive definite?

7*. Let V_n be the vector space of bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^3$. Let W_n be the subspace of V_n of maps T satisfying $T(u, v) = -T(v, u)$. Determine (and prove) formulas for the dimensions of V_n and W_n.

From Rotman:
9.53, 9.54, 9.55, 9.56, 9.58,

NOTE: The final exam is scheduled (by the registrar's office) for Wednesday Dec 21 at 4 pm. Place TBA.