MATH 593: Eleventh Homework Assignment: Bilinear Forms

Due Monday December 5, 2005

This week’s reading:

Please read the handout from Rotman, Advanced Modern Algebra, by Monday November 21.

1*. Consider linear transformations \( T_1 : V_1 \rightarrow W_1 \) and \( T_2 : V_2 \rightarrow W_2 \). Note that there is a naturally induced linear transformation \( T_1 \otimes T_2 : V_1 \otimes V_2 \rightarrow W_1 \otimes W_2 \).

a). If \( A_1 \) denotes the matrix of \( T_1 \) with respect to the bases \( \{v_1, \ldots, v_n\} \) and \( \{w_1, \ldots, w_m\} \) for \( V_1 \) and \( W_1 \) respectively, and \( A_2 \) denotes the matrix of \( T_2 \) with respect to the bases \( \{v'_1, \ldots, v'_p\} \) and \( \{w'_1, \ldots, w'_q\} \) for \( V_2 \) and \( W_2 \) respectively, describe the matrix for \( T_1 \otimes T_2 \) in terms of the bases \( \{v_i \otimes v'_j\} \) and \( \{w_k \otimes w'_l\} \) for \( V_1 \otimes V_2 \) and \( W_1 \otimes W_2 \) respectively. This is called the tensor product of the matrices \( A_1 \) and \( A_2 \) and denoted \( A_1 \otimes A_2 \).

b). Prove that the tensor product of two positive definite square symmetric matrices over \( \mathbb{R} \) is positive definite.

c). Let \( Q_1 \) and \( Q_2 \) be quadratic forms on finite dimensional real vector spaces \( V_1 \) and \( V_2 \) respectively. Show that there is a quadratic form on \( V_1 \otimes V_2 \) satisfying \( Q(v_1 \otimes v_2) = Q_1(v_1)Q_2(v_2) \).

d). Find the rank and signature of \( Q \) from part c in terms of those for \( Q_1 \) and \( Q_2 \).

2.* After an orthonormal change of basis (over \( \mathbb{R} \)), the quadratic form \( 2x_1x_2 + 2x_2x_3 + 6x_3x_4 \) becomes \( a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 \) for some \( a_i \).

a.) Compute explicitly the values \( a_1 + a_2 + a_3 + a_4 \) and \( a_1a_2a_3a_4 \) without computing the \( a_i \).

b). Is this quadratic form positive definite, negative definite, and/or non-degenerate?
3.* A real symmetric $4 \times 4$ matrix has characteristic polynomial $x^4 - dx + 12$ for some real number $d$. Is the corresponding quadratic form positive definite, negative definite, and/or non-degenerate? How many eigenvalues are positive?

4.* Let $\phi$ be an alternating bilinear form on an $n$-dimensional vector space $V$. Assume that the maximal dimension of a subspace $W \subset V$ such that $\phi(x, y) = 0$ for all $x, y$ in $W$ is zero is $m$ (such a subspace is called isotropic.) Find the rank of $\phi$.

5.* Let $A$ be a real matrix of any dimensions. Prove that $A^t A$ and $AA^t$ are positive semi-definite and have the same non-zero eigenvalues with the same multiplicities.

6.* Consider the real matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & a \end{pmatrix}$.

a). For each $a$, determine the rank and signature of a bilinear form with matrix $A$.

b). For which values of $a$ is $A$ diagonalizable?

c). For which values of $a$ is $A$ positive definite?

7*. Let $V_n$ be the vector space of bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^3$. Let $W_n$ be the subspace of $V_n$ of maps $T$ satisfying $T(u, v) = -T(v, u)$. Determine (and prove) formulas for the dimensions of $V_n$ and $W_n$.

From Rotman:
9.53, 9.54, 9.55, 9.56, 9.58,

NOTE: The final exam is scheduled (by the registrar’s office) for Wednesday Dec 21 at 4 pm. Place TBA.