This week’s reading: Handout from Artin’s book *Algebra*, sections 3, 4, and 5 of Chapter 7.

(QR) 1*. a) Let $V$ be a finite dimensional vector space over a field $F$ and let $A$ be a diagonalizable linear transformation of $V$. Prove that if $U$ is a subspace for which $AU \subset U$, then the restriction of $A$ to $U$ is also diagonalizable.

b). Let $A$ and $B$ be $n \times n$ matrices, diagonalizable over $F$. Prove that if $AB = BA$, then $A$ and $B$ are simultaneously diagonalizable, that is, there exists an invertible matrix $P$ such that $PAP^{-1}$ and $PBP^{-1}$ are both diagonal.

2*. Prove the following theorem: If $B$ is a bilinear form on a vector space $V$ over a field $k$, then the relationship of orthogonality it determines is symmetric if and only if $B$ is either a symmetric or an alternating bilinear form.

[Hint: Given three arbitrary vectors $x, y, z$ in $V$, consider the vector $w = B(x, y)z - B(x, z)y$. Consider now what it means for $w$ to be perpendicular to $x$ and for $x$ to be perpendicular to $w$. What happens when $x = y$?]

3*. Let $V$ be a real vector space, equipped with a positive definite symmetric bilinear form $f$. Prove the triangle and Schwartz inequalities, that is:

a). Show that $f(v + w, v + w)^{1/2} \leq f(v, v)^{1/2} + f(w, w)^{1/2}$.

b). $f(v, w) \leq f(v, v)^{1/2} f(w, w)^{1/2}$.

4*. Let $V$ be a finite dimensional complex vector space, and let $h$ be a Hermitian form on $V$.

a). Carefully state and prove a theorem telling how the matrix of $h$ with respect to a fixed basis transforms when we change basis.

b). Show that the relation you discovered in part a is an equivalence relation on the set of hermitian (ie, conjugate symmetric) complex $n \times n$ matrices. If $A$ and $B$ are equivalent hermitian matrices, show that they represent the same hermitian form in different bases.

c). Prove the hermitian analog of Sylvester’s Theorem: If $V$ is a finite dimensional complex vector space with hermitian form $h$, then there exists a basis for $V$ such that the corresponding matrix is

\[
\begin{pmatrix}
I_p & 0 & 0 \\
0 & -I_q & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

where $I_p$ and $I_q$ are identity matrices, and the 0 are block zero matrices.
d). Prove that the rank and signature \((p, q)\) of a hermitian form is well-defined.

5*. a). Let \(V\) be a real vector space of finite dimension \(n\) with positive definite symmetric bilinear form \(f\). Show that the group of isometries of \((V, f)\) (those linear transformations \(T : V \to V\) such that \(f(Tx, Ty) = f(x, y)\) for all \(x, y \in V\)) is isomorphic to the group \(O_n(\mathbb{R})\) of invertible \(n \times n\) real matrices \(P\) such that \(P^{-1} = P^{tr}\).

b). Let \(V\) be a complex vector space of finite dimension \(n\) with positive definite hermitian form \(h\). Show that the group of isometries of \((V, h)\) (those linear transformations satisfying \(T : V \to V\) such that \(h(Tx, Ty) = h(x, y)\) for all \(x, y \in V\)) is isomorphic to the group \(U_n\) of invertible \(n \times n\) matrices \(P\) such that \(P^{-1} = P^*\), where \(P^*\) denotes the transpose of the matrix of obtained from \(P\) by conjugating all its entries.

6*. Let \((V, f)\) be a vector space of finite dimension \(2n\) over a field \(k\) of characteristic not 2, together with a non-degenerate alternating form \(f\). Show that the set of all linear transformations \(T : V \to V\) which preserve \(f\) forms a group under composition.

a). Show that the group of part a is isomorphic to the symplectic group \(Sp_{2n}(k)\), namely the group of matrices \(A\) in \(GL_{2n}(k)\) satisfying \(A^tJA = J\) where is the standard skew-symmetric matrix

\[
J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.
\]

b). Deduce an analogous theorem about real symmetric matrices.

From Artin:
1.2, 1.3, 3.4, 4.11, 4.10, 4.17 .

NOTE: The final exam is scheduled (by the registrar’s office) for Wednesday Dec 21 at 4 pm. Place TBA.