MATH 593: Twelth and Final Homework Assignment: Symmetric and Hermitian Forms

Due Friday December 16, 2005 at 1 pm.

This week's reading: Handout from Artin's book *Algebra*, sections 3, 4, and 5 of Chapter 7.

(QR) 1*. a) Let V be a finite dimensional vector space over a field F and let A be a diagonalizable linear transformation of V. Prove that if U is a subspace for which $AU \subset U$, then the restriction of A to U is also diagonalizable.

b). Let A and B be $n \times n$ matrices, diagonalizable over F. Prove that if $AB = BA$, then A and B are simultaneously diagonalizable, that is, there exists an invertible matrix P such that PAP^{-1} and PBP^{-1} are both diagonal.

2*. Prove the following theorem: *If B is a bilinear form on a vector space V over a field k, then the relationship of orthogonality it determines is symmetric if and only if B is either a symmetric or an alternating bilinear form.*

[Hint: Given three arbitrary vectors x, y, z in V, consider the vector $w = B(x, y)z - B(x, z)y$. Consider now what it means for w to be perpendicular to x and for x to be perpendicular to w. What happens when $x = y$?]

3*. Let V be a real vector space, equipped with a positive definite symmetric bilinear form f. Prove the triangle and Schwartz inequalities, that is:

a). Show that $f(v + w, v + w)^{1/2} \le f(v, v)^{1/2} + f(w, w)^{1/2}$.

b). $f(v, w) \le f(v, v)^{1/2} f(w, w)^{1/2}$.

4*. Let V be a finite dimensional complex vector space, and let h be a Hermitian form on V.

a). Carefully state and prove a theorem telling how the matrix of h with respect to a fixed basis transforms when we change basis.

b). Show that the relation you discovered in part a is an equivalence relation on the set of hermitian (ie, conjugate symmetric) complex $n \times n$ matrices. If A and B are equivalent hermitian matrices, show that they represent the same hermitian form in different bases.

c). Prove the hermitian analog of Sylvester's Theorem: If V is a finite dimensional complex vector space with hermitian form h, then there exists a basis for V such that the corresponding matrix is $\begin{pmatrix} I_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$, where I_p and I_q are identity matrices, and the $\mathbf{0}$ are block zero matrices.

d). Prove that the rank and signature (p, q) of a hermitian form is well-defined.

5*. a). Let V be a real vector space of finite dimension n with positive definite symmetric bilinear form f. Show that the group of isometries of (V, f) (those linear tranformations $T : V \to V$ such that $f(Tx, Ty) = f(x, y)$ for all $x, y \in V$) is isomorphic to the group $O_n(\mathbb{R})$ of invertible $n \times n$ real matrices P such that $P^{-1} = P^{tr}$.

b). Let V be a complex vector space of finite dimension n with positive definite hermitian form h. Show that the group of isometries of (V, h) (those linear tranformations satisfying $T : V \to V$ such that $h(Tx, Ty) = h(x, y)$ for all $x, y \in V$) is isomorphic to the group U_n of invertible $n \times n$ matrices P such that $P^{-1} = P^*$, where P^* denotes the transpose of the matrix of obtained from P by conjugating all its entries.

6*. Let (V, f) be a vector space of finite dimension $2n$ over a field k of characteristic not 2, together with a non-degenerate alternating form f.

a). Show that the set of all linear transformations $T : V \to V$ which preserve f forms a group under composition.

b). Show that the group of part a is isomorphic to the symplectic group $Sp_{2n}(k)$, namely the group of matrices A in $GL_{2n}(k)$ satisfying $A^{tr}JA = J$ where is the standard skew-symmetric matrix $J \begin{pmatrix} \mathbf{0} & -I_n \\ I_n & \mathbf{0} \end{pmatrix}$.

7*. a). Prove that an n by n matrix A over the complex numbers is a positive definite hermitian matrix if and only if $A = P^*P$ for some invertible matrix P, where P^* denotes the transpose of the matrix of obtained from P by conjugating all its entries.

b). Deduce an analogous theorem about real symmetric matrices.

From Artin:

1.2, 1.3, 3.4, 4.11, 4.10, 4.17 .

NOTE: The final exam is scheduled (by the registrar's office) for Wednesday Dec 21 at 4 pm. Place TBA.