Math 593: Homework 6

October 24, 2014

1. Let A be a $n \times n$ matrix of integers. Prove that the cokernel is finite if and only if the determinant Δ is non-zero. In this case, prove that the cardinality of the cokernel is $|\Delta|$.

2. Semi-simple vs. Diagonalizable. A module over a ring R is semi-simple if it is a direct sum of simple modules. Define a linear operator on a finite dimensional vector space to be semi-simple if the corresponding $k[x]$-module is semi-simple.

a). Describe all simple $k[x]$-modules.

b). Show that if T is diagonalizable, then it is semi-simple. Show that the converse holds if k is algebraically closed.

c). Show that if k fails to be algebraically closed, then there exists a semi simple but not diagonalizable linear operator on a finite dimensional vector space over k.

d). Is rotation through $\pi/2$ a semi-simple transformation of $\mathbb{R}^2$? Explain why it can't be diagonalizable using a *geometric* argument about eigenvectors.

e). Show that T is semi-simple if its minimal polynomial has no repeated factors, and diagonalizable if those distinct factors are all *linear*.

f). Show that a linear operator of finite order (meaning T^d is the identity, for some d) is diagonalizable over $\mathbb{C}$.

g). Define the λ-eigenspace of an operator T on V to be the set of all vectors on which T acts by multiplication by λ. Prove that this is a subspace and show that T is diagonalizable if and only if the sum of the eigenspaces (over all eigenvalues λ) is all of V.

3. Simultaneous Diagonalizabilty. a). Show that if S and T are commuting linear operators, then any eigenspace for T is invariant under S.

b). Show that if two diagonalizable matrices commute, then they are *simultaneously diagonalizable:* there is a *single* basis in which *both* are represented by diagonal matrices.

4. Jordan-Chevelley Decomposition. Let T be a linear operator on a finite dimensional vector space V over an algebraically closed field k.

a). Using Jordan decomposition, show that $T = N + D$ where N is a nilpotent operator and D is diagonalizable, and N and D commute.

b). Show that D is a polynomial in T. [Hint: If m is the minimal polynomial of T, use the Chinese remainder theorem *for the ring* $k[x]/(m)$ to find some polynomial $g \in k[x]$ which acts by multiplication by λ_i on each submodule of the form $k[x]/(x-\lambda_i)^{a_i}$ in the elementary divisor decomposition of the $k[x]$-module V.]

c). Show that if two nilpotent (respectively diagonalizable) operators commute, then their sum is nilpotent (respectively, diagonalizable).

d). Show that if $T = N' + D'$ where N' is nilpotent, D' is diagonalizable and N' and D' commute, then $N = N'$ and $D = D'$.

5. Trace. Let A be an $n \times n$ matrix over a field k. Define the *trace* of A to be the sum of the diagonal entries: $tr(A) = \sum_{i=1}^{n} a_{ii}$.

a). Directly from the definition, show that similar matrices have the same trace by using the fact that an invertible $n \times n$ matrix P is a product of elementary matrices (corresponding to elementary row/column ops).

b). Show that the trace of A is $-a_{n-1}$ where a_{n-1} is the coefficient of x^{n-1} in the characteristic polynomial of A. [Use the invariant factor decomposition of the corresponding $k[x]$-module.]

c). Show that the trace of A is the sum of the eigenvalues of A (viewing the eigenvalues, if necessary, in $\overline{k}$, the algebraic closure of k).

d). Show that the trace is a linear mapping from the space of $n \times n$ matrices to k. Is tr(AB) = tr(A)tr(B)?

e). Show that the trace of A^d is the sum of the d-th powers of the eigenvalues of A. [Hint: first view the matrix over $\overline{k}$.]

f). Show that if k has characteristic zero, then A is nilpotent if and only if A^d has trace zero for all $d \in \mathbb{N}$. Give a counterexample to this statement when k has characteristic $p > 0$.

6. Some QR problems.

a). Let P be an $n \times n$ matrix with entries in a field such that $P^2 = P$. What are the possible values of the trace of P?

b) Let G be a finite abelian group in which every element has order dividing 63 and in which there are 108 elements of order exactly 63. Determine all possibilities for the structure of G (up to isomorphism).