Math 614: Problem Set 2

Due Tuesday October 8, 2019

1. Let $\mathcal{C}$ be a category. For a fixed object $X \in Ob(\mathcal{C})$, define a functor $h_X : \mathcal{C} \rightarrow Set$ sending $Y \mapsto Mor_{\mathcal{C}}(X, Y)$.

   (a) For the category $\mathbb{Z}$-mod, and the object $X = \mathbb{Z}/(12)$, describe $h_X(\mathbb{Z}/(24) \oplus \mathbb{Z}/(7) \oplus \mathbb{Q})$ explicitly. In particular, what is its cardinality?

   (b) For each prime ideal $P$ of $\mathbb{R}$, let $\mathcal{P}$ be a subring of $\mathbb{R}$.

   (c) Let $\mathcal{P}$ be a finite group, and suppose $\mathcal{P}$ acts on a commutative ring $S$ by ring automorphisms. Let $S^G = \{ f \in S \mid g \cdot f = f \}$.

2. For each prime ideal $P$ of a ring $R$, define $P^{(n)}$ as $P^n R_P \cap R$. Let $T = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables, and let $S = T/(\sum_{i=1}^{n} x_i y_i)$. Compute the following

   (a) For $P = (x_1, \ldots, x_n, y_1, \ldots, y_n) \subset T$, compute $P^{(4)}$.

   (b) For $P = (x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) \subset T$, compute $P^{(3)}$.

   (c) For $P = (x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) \subset S$, compute $P^{(2)}$. Here we abuse notation, using elements of $T$ to represent their classes in $S$.

3. Let $R$ be a subring of $S$. Suppose that there is an $R$-module map $\theta : S \rightarrow R$ such that $\theta(1) = 1$.

   (a) Show that for every ideal of $R$, $IS \cap R = I$.

   (b) Show that the map $Spec S \rightarrow Spec R$ is surjective.

   (c) Let $G$ be a finite group, and suppose $G$ acts on a commutative ring $S$ by ring automorphisms. Let $S^G = \{ f \in S \mid g \cdot f = f \}$. Assuming that $|G|$ is a unit in $S$, prove or disprove that induced map of Spectra is surjective.

   (d) (*) Same as (c) without the assumption that $|G|$ is a unit in $S$.

4. Let $\zeta \in \mathbb{C}$ be a primitive $n$-th root of unity. Define an action of the cyclic group $C_n = \{ \zeta^i \mid i \in \mathbb{N} \} \subset \mathbb{C}^\times$ by $\mathbb{C}$-algebra automorphisms on the polynomial ring $\mathbb{C}[x, y]$ by declaring $\zeta \cdot f(x, y) = f(\zeta^{-1} x, \zeta^{-1} y)$.
(a) Describe the induced action of $C_n$ on $\text{maxSpec } \mathbb{C}[x, y]$ by explicitly explaining the action on an arbitrary maximal ideal. Describe the corresponding action on $\mathbb{C}^2$ induced by the Nullstellensatz.

(b) Describe the ring of invariants $\mathbb{C}[x, y]^{C_n} = \{ f \in \mathbb{C}[x, y] \mid g \cdot f = f \ \forall g \in C_n \}$ as a subring of $\mathbb{C}[x, y]$. [HINT: It may be helpful to think about the degrees of polynomials.]

(c) Show that $\mathbb{C}[x, y]^{C_n}$ is finitely generated over $\mathbb{C}$ by $n + 1$ elements.

(d) Find a presentation for $\mathbb{C}[x, y]^{C_n}$ in the case $n = 2$.

(e) * For the ring inclusion $\mathbb{C}[x, y]^{C_n} \subset \mathbb{C}[x, y]$, show that the induced map $\text{maxSpec } \mathbb{C}[x, y] \to \text{maxSpec } \mathbb{C}[x, y]^{C_n}$ can be identified with the quotient map $\mathbb{C}^2 \to V$, where $V$ is the set of orbits of the action of $C_n$ on $\mathbb{C}^2$ described in (a).

5.* For any vector space $V$ over $K$, let $P(V)$ denote the set of all one-dimensional subspaces of $V$. When $V$ is the space of column vectors $K^{n+1}$, a point $P$ in $P(V)$ can be represented by a (column) vector $v$ spanning $P$; this is well-defined only up to non-zero scalar multiple. Define a map

$$\Sigma : P(K^{n+1}) \times P(K^{m+1}) \to P(K^{(n+1) \times (m+1)})$$

sending $(v, w) \mapsto vw^t$, the (one-dimensional space spanned by the) matrix product of the column vector $v$ with the row vector $w^t$.

1. Prove that $\Sigma$ is well-defined and injective.

2. Find polynomials in $(m + 1)(n + 1)$ variables which vanish on a point $P$ in $P(K^{(n+1) \times (m+1)})$ if and only if $P$ is in the image of $\Sigma$.

3. Let $\pi : \text{im}(\Sigma) \to P(K^{n+1})$ (respectively, $\psi : \text{im}(\Sigma) \to P(K^{m+1})$) be defined by taking a representative $Q \in K^{(n+1) \times (m+1)}$ to any of its columns (respectively rows). Show that these maps are well-defined and that $\text{im}(\Sigma) \xrightarrow{\pi, \psi} P(K^{n+1}) \times P(K^{m+1})$ defines an inverse map to $\Sigma$.

6.* The ring of complex analytic germs in $d$ variables, denoted $\mathbb{C}\{z_1, \ldots, z_d\}$, is the subring of $\mathbb{C}\{z_1, \ldots, z_d\}$ consisting of power series that converge on some ball containing the origin.

- A Weierstrass polynomial of degree $t$ in $z_d$ is a function of the form $z^t + f_{t-1}z^{t-1} + \cdots + f_0$ with $f_0, \ldots, f_{t-1} \in \mathbb{C}\{z_1, \ldots, z_{d-1}\}$.

- The Weierstrass preparation theorem says that: If $f \in \mathbb{C}\{z_1, \ldots, z_d\}$, satisfies $f(0, \ldots, 0) = 0$, and $f(0, \ldots, 0, z_d) \neq 0$, then there is some unit $g \in \mathbb{C}\{z_1, \ldots, z_d\}$, and Weierstrass polynomial $h$ in $z_d$ such that $f = gh$.

- The Weierstrass division theorem says that if $h$ is a Weierstrass polynomial of degree $t$ in $z_d$, and $f \in \mathbb{C}\{z_1, \ldots, z_d\}$, then $f = ph + q$ for some $p \in \mathbb{C}\{z_1, \ldots, z_d\}$, and $q \in \mathbb{C}\{z_1, \ldots, z_{d-1}\}[z_d]$ of degree less than $t$ in $z_d$.

Use the Weierstrass preparation theorem and Weierstrass division theorem to show that $\mathbb{C}\{z_1, \ldots, z_d\}$ is Noetherian.