1. Let \( \mathcal{C} \) be a category. For a fixed object \( X \in \text{Ob}(\mathcal{C}) \), define a functor \( h_X : \mathcal{C} \to \text{Set} \) sending \( Y \to \text{Mor}_\mathcal{C}(X, Y) \).

1. For the category \( \mathbb{Z}\text{-mod} \), and the object \( X = \mathbb{Z}/(12) \), describe \( h_X(\mathbb{Z}/(24) \oplus \mathbb{Z}/(7) \oplus \mathbb{Q}) \) explicitly. In particular, what is its cardinality?

2. Given a morphism \( Y \xrightarrow{g} X \), define, for each \( Z \in \mathcal{C} \), a map of sets \( \xi \mathcal{T}_Z \) by

\[
\xi \mathcal{T}_Z(X) : \text{Mor}_\mathcal{C}(X, Z) \to \text{Mor}_\mathcal{C}(Y, Z) \quad f \to f \circ g.
\]

Verify that \( \xi \mathcal{T}_Z \) defines a natural transformation from \( h_X \) to \( h_Y \).

3. Show that there is a bijection between \( \text{Mor}(Y, X) \) and the set of all natural transformations from \( h_X \) to \( h_Y \).

4. A natural transformation \( T \) between two functors \( F, G : \mathcal{D} \to \mathcal{D}' \) is an isomorphism of functors, if, for every object \( X \in \mathcal{D} \), the morphism \( T(X) : F(X) \to G(X) \) is an isomorphism in \( \mathcal{D}' \). Show that \( h_X \) and \( h_Y \) are isomorphic functors if and only if \( X \) and \( Y \) are isomorphic objects.

5. Show that if a covariant functor is representable, then the object representing it is unique, up to isomorphism.

2. For each prime ideal \( P \) of a ring \( R \), define \( P^{(n)} \) as \( P^n R_P \cap R \). Let \( T = K[x_1, \ldots, x_n, y_1, \ldots, y_n] \) be the polynomial ring in \( 2n \) variables, and let \( S = T/(\sum_{i=1}^{n} x_i y_i) \). Compute the following

(a) For \( P = \langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \subset T \), compute \( P^{(4)} \).

(b) For \( P = \langle x_1, \ldots, x_n, y_1, \ldots, y_{n-1} \rangle \subset T \), compute \( P^{(3)} \).

(c) For \( P = \langle x_1, \ldots, x_n, y_1, \ldots, y_{n-1} \rangle \subset S \), compute \( P^{(2)} \). Here we abuse notation, using elements of \( T \) to represent their classes in \( S \).

3. Let \( R \) be a subring of \( S \). Suppose that there is an \( R \)-module map \( \theta : S \to R \) such that \( \theta(1) = 1 \).

(a) Show that for every ideal of \( R \), \( IS \cap R = I \).

(b) Show that the map \( \text{Spec} S \to \text{Spec} R \) is surjective.

(c) Let \( G \) be a finite group, and suppose \( G \) acts on a commutative ring \( S \) by ring automorphisms. Let \( S^G = \{ f \in S \mid g \cdot f = f \} \). Assuming that \( |G| \) is a unit in \( S \), prove or disprove that induced map of Spectra is surjective.

(d) (*) Same as (c) without the assumption that \( |G| \) is a unit in \( S \).

4. Let \( \zeta \in \mathbb{C} \) be a primitive \( n \)-th root of unity. Define an action of the cyclic group \( C_n = \{ \zeta^i \mid i \in \mathbb{N} \} \subset \mathbb{C}^* \) by \( \mathbb{C} \)-algebra automorphisms on the polynomial ring \( \mathbb{C}[x, y] \) by declaring \( \zeta \cdot f(x, y) = f(\zeta^{-1} x, \zeta^{-1} y) \).
(a) Describe the induced action of $C_n$ on $	ext{maxSpec } \mathbb{C}[x, y]$ by explicitly explaining the action on an arbitrary maximal ideal. Describe the corresponding action on $\mathbb{C}^2$ induced by the Nullstellensatz.

(b) Describe the ring of invariants $\mathbb{C}[x, y]^{C_n} = \{ f \in \mathbb{C}[x, y] \mid g \cdot f = f \ \forall g \in C_n \}$ as a subring of $\mathbb{C}[x, y]$. [HINT: It may be helpful to think about the degrees of polynomials.]

(c) Show that $\mathbb{C}[x, y]^{C_n}$ is finitely generated over $\mathbb{C}$ by $n + 1$ elements.

(d) Find a presentation for $\mathbb{C}[x, y]^{C_n}$ in the case $n = 2$.

(e) * For the ring inclusion $\mathbb{C}[x, y]^{C_n} \subset \mathbb{C}[x, y]$, show that the induced map maxSpec $\mathbb{C}[x, y] \to$ maxSpec $\mathbb{C}[x, y]^{C_n}$ can be identified with the quotient map $\mathbb{C}^2 \to V$, where $V$ is the set of orbits of the action of $C_n$ on $\mathbb{C}^2$ described in (a).

5.* For any vector space $V$ over $K$, let $\mathbb{P}(V)$ denote the set of all one-dimensional subspaces of $V$. When $V$ is the space of column vectors $K^{n+1}$, a point $P$ in $\mathbb{P}(V)$ can be represented by a (column) vector $v$ spanning $P$; this is well-defined only up to non-zero scalar multiple. Define a map

$$\Sigma : \mathbb{P}(K^{n+1}) \times \mathbb{P}(K^{m+1}) \to \mathbb{P}(K^{(n+1)\times(m+1)})$$

sending $(v, w) \to v w^t$, the (one-dimensional space spanned by the) matrix product of the column vector $v$ with the row vector $w^t$.

1. Prove that $\Sigma$ is well-defined and injective.

2. Find polynomials in $(m + 1)(n + 1)$ variables which vanish on a point $P$ in $\mathbb{P}(K^{(n+1)\times(m+1)})$ if and only if $P$ is in the image of $\Sigma$.

3. Let $\pi : \text{im}(\Sigma) \to \mathbb{P}(K^{n+1})$ (respectively, $\psi : \text{im}(\Sigma) \to \mathbb{P}(K^{m+1})$) be defined by taking a representative $Q \in K^{(n+1)\times(m+1)}$ to any of its columns (respectively rows). Show that these maps are well-defined and that $\text{im}(\Sigma) \xrightarrow{\pi, \psi} \mathbb{P}(K^{n+1}) \times \mathbb{P}(K^{m+1})$ defines an inverse map to $\Sigma$.

6.* The ring of complex analytic germs in $d$ variables, denoted $\mathbb{C}\{z_1, \ldots, z_d\}$, is the subring of $\mathbb{C}\{[z_1, \ldots, z_d]\}$ consisting of power series that converge on some ball containing the origin.

- A Weierstrass polynomial of degree $t$ in $z_d$ is a function of the form $z^t + f_{t-1}z^{t-1} + \cdots + f_0$ with $f_0, \ldots, f_{t-1} \in \mathbb{C}\{z_1, \ldots, z_{d-1}\}$.

- The Weierstrass preparation theorem says that: If $f \in \mathbb{C}\{z_1, \ldots, z_d\}$, satisfies $f(0, \ldots, 0) = 0$, and $f(0, \ldots, 0, z_d) \neq 0$, then there is some unit $g \in \mathbb{C}\{z_1, \ldots, z_d\}$, and Weierstrass polynomial $h$ in $z_d$ such that $f = gh$.

- The Weierstrass division theorem says that if $h$ is a Weierstrass polynomial of degree $t$ in $z_d$, and $f \in \mathbb{C}\{z_1, \ldots, z_d\}$, then $f = ph + q$ for some $p \in \mathbb{C}\{z_1, \ldots, z_d\}$, and $q \in \mathbb{C}\{z_1, \ldots, z_{d-1}\}[z_d]$ of degree less than $t$ in $z_d$.

Use the Weierstrass preparation theorem and Weierstrass division theorem to show that $\mathbb{C}\{z_1, \ldots, z_d\}$ is Noetherian.