1. Let $S$ be an $R$-algebra, let $M$ be an $R$-module and let $N$ be an $S$-module.

1. Show that there is a natural $S$-module isomorphism $\text{Hom}_R(M, N) \cong \text{Hom}_S(S \otimes_R M, N)$.
2. Show that $S \otimes_R M$ represents the functor $\text{Hom}_R(M, -)$ from $S$-mod to $S$-mod.

2. Let $R$ be an $\mathbb{F}_p$-algebra, where $p > 0$ is prime.

1. Show that the map $F : R \to R$ sending $r \mapsto r^p$ is a ring homomorphism.
2. Show that $F$ is module finite if and only if $F$ is algebra finite.
3. Show that the induced map on $\text{Spec}$ for the map $F$ in (a) is the identity map.
4. In the case that $R = \mathbb{F}_p[x_1, \ldots, x_d]$, prove that $R$ is free as an $R$-module when viewed via restriction of scalars from $F$, and find its rank.

3. Let $R$ be a $K$-algebra, where $K$ is a field. Let $J, I_1, \ldots, I_t$ be ideals of $R$. Suppose that $J \subset I_1 \cup I_2 \cup \cdots \cup I_t$.

1. Assuming $K$ is infinite, show that $J \subset I_j$ for some $j = 1, 2, \ldots, t - 1$ or $t$.
2. Without assuming $K$ is infinite, but assuming $I_1, \ldots, I_{t-2}$ are prime, show that $J \subset I_j$ for some $j = 1, 2, \ldots, t - 1$ or $t$.

4. Let $R$ be a ring and let $P_1, P_2, \ldots, P_t$ be prime ideals.

1. Let $U = R \setminus (\bigcup_{i=1}^t P_i)$. Show that $U$ is a multiplicative system.
2. Show that $U^{-1}R$ has finitely many maximal ideals and describe them explicitly.
3. Fix a natural number $t > 1$ and $d_1 < d_2 < \ldots < d_t$. Construct Noetherian domain $S$ which admits exactly $t$ maximal ideals of heights $d_1, d_2, \ldots, d_t$, respectively.

5. Consider a system of $m$ linear equations in $n$ unknowns over a commutative ring $R$. Prove that the equations have a solution in $R$ if and only if for every maximal ideal of $R$ the corresponding system in which the coefficients are replaced by their images in $R_m$ has a solution in $R_m$.

6.* Consider a doubly indexed set of variables $\{x_{ij} | i \leq j, i, j \in \mathbb{N}\}$. Let $S$ be the polynomial ring they generate over $\mathbb{C}$, so $S = \mathbb{C}[x_{11}, x_{12}, x_{22}, x_{13}, x_{23}, x_{33}, \ldots]$. For each fixed $j$, let $P_j$ be the prime ideal generated by $\{x_{1j}, x_{2j}, \ldots, x_{jj}\}$. Let $U = S \setminus \bigcup_{j=1}^\infty P_j$. Let $R = U^{-1}S$. 
1. Show that if an ideal $I \subset S$ is contained in $\bigcup_{n=1}^{\infty} P_n$, then $I \subset P_n$ for some $n$. [Hint: for $f \in I$, consider the (non empty, finite) set $Q(f) := \{i \in \mathbb{N} \mid f \in P_i \mathbb{R}\}$. Show we’re done unless $\forall f \in I, \exists g \in I$ such that $Q(f) \cap Q(g) = \emptyset$. Now look at $f + x_m^d g$ (which is in $I$) for well-chosen $m \in Q(g)$ and $d \gg 0$.]

2. Show that $R$ has chains of primes of arbitrarily long length.

3. Prove that the localization of $R$ at any maximal ideal is Noetherian.

4. Prove that $R$ is Noetherian but has infinite Krull dimension. [Hint: Problem 5 on Problem Set 3 might be helpful.]