1. A family of elliptic curves. For $\lambda \in k$, let $C_\lambda$ denote the plane projective cubic curve defined by $x^3 + y^3 + z^3 + \lambda(x + y + z)^3$. Construct a family of varieties parametrized by the affine line whose members are exactly the $C_\lambda$. Explicitly describe the locus of points in your parameter space $A^1$ parametrizing those $C_\lambda$ that are smooth.

2. Maps on Tangent Spaces. Say that $V$ and $W$ are closed subvarieties of $A^n$ and $A^m$ respectively, and let $F : V \rightarrow W$ be a regular map. Prove that the induced linear map on tangent spaces $d_pF : T_pV \rightarrow T_{F(p)}W$ at a point $p \in V$ is induced by (describe how, and in what basis!) by the Jacobian matrix (at $p$) of the coordinate functions describing $F$.

3. Birational equivalence of varieties. Define two varieties to be birationally equivalent if they are isomorphic on dense open subsets. That is, $W$ and $V$ are birationally equivalent if there exist dominant regular maps $\phi : U \rightarrow W$ and $\psi : U' \rightarrow V$, where $U$ is a dense open subset of $V$ and $U'$ is a dense open subset of $W$, and some dense open subset of $W$ (respectively $V$) where the composition $\phi \circ \psi$ (respectively $\psi \circ \phi$) is the identity map on $W$ (respectively $V$).

   a). Prove that $V$ and $W$ are birationally equivalent if and only if their function fields $k(V)$ and $k(W)$ are isomorphic as field extensions of $k$.

   b). Show that birationally equivalent varieties have the same dimension.

   c). Show that every variety is birationally equivalent to a hypersurface. [This may involve quoting some facts from commutative algebra, and is much easier if you’ve done all the assigned reading from Shafarevich.]

4. Singular set of Projective Varieties. Let $V \subset P^n$ be a projective variety and let $\tilde{V} \subset k^{n+1}$ be the affine cone over it.

   a). Prove that $V$ is smooth if and only if $\tilde{V}$ has (at worst) an isolated singular point at the origin.

   b). Prove that $\tilde{V}$ is always singular at the origin unless $V$ is a linear variety in $P^n$.

   c). If the homogeneous ideal of $V$ is generated by homogeneous polynomials $F_1, F_2, \ldots, F_m$ and $V$ has pure codimension $c$, show that $\text{Sing } V = V \cap \forall(c \times c \text{ minors } \frac{\partial F_i}{\partial x_j})$.

5. Universal Hypersurface. Consider the variety $X$ defined by the bihomogeneous polynomial $\sum a_I x^I$ in $P(Sym^d(V^*)) \times P(V)$, where $x^I$ denotes $x_0^{i_1} \ldots x_n^{i_n}$ for $x_0, \ldots, x_d$ coordinates for $P(V)$ and $a_I$ the corresponding coordinates on $P(Sym^d(V^*))$.

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1 in the technical sense of a surjective morphism of varieties
a). Explain how this allows us to think of the set of hypersurfaces in $\mathbb{P}(V)$ as a family in the technical sense of the word, parametrized by $\mathbb{P}(Sym^d(V^*))$.

b). Prove that $\mathcal{X}$ is an irreducible projective variety by considering one of the two obvious projections.

c). Consider the subset $\Sigma \subset \mathcal{X}$ consisting of pairs $(H,p) \in \mathbb{P}(Sym^d(V^*)) \times \mathbb{P}(V)$ where $p$ is a singular point on the hypersurface in $\mathbb{P}(V)$ corresponding to $H$. Prove that $\Sigma$ is a proper closed subset of $\mathcal{X}$ by giving some explicit defining equations.

d). What is the geometric significance of the image of $\Sigma$ under the projection to $\mathbb{P}(Sym^d(V^*))$? Can the image of $\Sigma$ be all of $\mathbb{P}(Sym^d(V^*))$?

6. Bertini’s Theorem. a). Let $V = \mathbb{V}(F)$ be a smooth irreducible hypersurface in $\mathbb{P}^n$, not contained in any hyperplane $\mathbb{P}^{n-1}$. Prove that a hyperplane section $V \cap H$ is smooth at $p$ if and only if the hyperplane $H$ is not tangent to $V$ at $p$.

b). Prove that the set of smooth hyperplane sections of $V$ is parametrized by a non-empty Zariski open subset of the variety of hyperplanes in $\mathbb{P}^n$. (This is called Bertini’s theorem, and is true even if $V$ is not a hypersurface.)

7. Gauss Map. Let $X$ be a smooth irreducible projective variety of dimension $d$ in $\mathbb{P}(V) = \mathbb{P}^n$. Define a map $\rho : X \rightarrow G_{d+1}(V)$ sending the point $p$ to the projective tangent space to $X$ at $p$.

a). Explain why there is a natural identification of $G_{d+1}(V)$ with $G_{n-d}(V^*)$.

b). Prove that $\rho$ is a morphism of varieties. What is going on when two points have the same image under $\rho$?

c). Let $X = \mathbb{V}(x^d + y^d + z^d)$ in $\mathbb{P}^2$. Compute $\rho$ explicitly. What is the dimension of the image? Compute the image explicitly in the cases where $d = 1, 2, \text{ and } 3$.

8. The Tangent Cone. a). Read page 95 of Shafarevich (section II 1.5). Accept the fact that the tangent cone of any variety at $p$ has the same dimension as $V$ at $p$. Show that if $p$ is a smooth point of $V$, then the tangent cone to $V$ at $p$ agrees with the tangent space to $V$ at $p$.

b). Prove that if $V$ is a plane curve, then the tangent cone to $V$ is a union of lines, and find the tangent cones at the origin for each of the plane curves below. Sketch each curve and its tangent cone.

$$y^2 - x^2 - x^3 = 0, \quad xy - y^4 = 0.$$ 

c). How do you expect the (complex points of the) curve defined by the equation $x^3 y - y^3 x - xy^{17} + xy^{203}$ to look in a tiny $\epsilon$-neighborhood of the origin (using the Euclidean topology)? What about $x^2 y + xy^2 - x^4 - y^4 = 0$?

d). Fix a finite set of scalars $\{m_1, \ldots, m_k\}$. Show that there exists an irreducible plane curve $C$ passing through a point $p$ which has $k$ branches passing through $p$ with slopes $m_1, \ldots, m_k$, respectively.

e). Find the tangent cone to the curve $\mathbb{V}(x^3 - y^2)$ at the origin. Why would it be better to think of this as a “scheme”? What scheme should it be?

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2The image is called the dual curve.

3Prove it for extra credit.