Math 631: Problem Set 5

Due Friday October 11, 2013

1. **Closed sets of** \( \mathbb{P}^n \times \mathbb{P}^m \). Recall that the closed sets in \( \mathbb{P}^n \times \mathbb{P}^m \) are given by bihomogeneous polynomials (see Shaf 5.1).

   a). Let \( X \) be a hyperplane section of the Segre variety \( \Sigma_{nm} \) in \( \mathbb{P}^{(m+1)(n+1)−1} \), that is, the intersection of a hyperplane in \( \mathbb{P}^{(m+1)(n+1)−1} \) with the image of \( \mathbb{P}^n \times \mathbb{P}^m \) under the Segre map \( \sigma_{nm} \). Explicitly describe the bihomogeneous polynomials that define the corresponding closed subset of \( \mathbb{P}^n \times \mathbb{P}^m \).

   b). Let \( X = \mathbb{V}(x_0^2x_1^2y_1y_0 + x_1^5y_1^2 + x_0^5y_1^2 + x_0x_1y_0^2) \) be a closed set in \( \mathbb{P}^1 \times \mathbb{P}^1 \) (where homogeneous coordinates are \( x_0 : x_1 \) on first copy and \( y_0 : y_1 \) on second copy of \( \mathbb{P}^1 \)). Find defining equations for the image of \( X \) under the (restriction of) the segre embedding \( \sigma_{1,1} \) in \( \mathbb{P}^3 \), with coordinates \( z_{00}, z_{01}, z_{10}, z_{11} \).

2. **Hyperplane through general points.** Fix any \( n \) points in \( \mathbb{P}^n \). Prove that there is a hyperplane containing these \( n \) points, and that if the \( n \) points are in “general position,” then there is a unique hyperplane containing them all. Explain the meaning of “general position” in the context of this problem. For example, when \( n \) is two, it is clear that two points determine a line; general position here means the points are distinct.\(^1\)

3. **Hypersurfaces through points.**
   
   a). Fix a point \( P \in \mathbb{P}(V) = \mathbb{P}^n \). Show that the set of hypersurfaces of degree \( d \) in \( \mathbb{P}(V) \) passing through \( P \) is naturally parametrized by a hyperplane in the variety \( \mathbb{P}(\text{Sym}^d(V^*)) \) parametrizing all degree \( d \) hypersurfaces in \( \mathbb{P}(V) \).

   b). Fix a natural number \( d \). Find \( q \) such that following sentence is true: “Through \( q \) general points in the projective plane, there passes a uniquely determined curve (ie, hypersurface in \( \mathbb{P}^2 \)) of degree \( d \).”

4. **Family of Degenerate Conics.** For this problem assume the field does not have characteristic 2.

   a). Show that the subset of degenerate conics in \( \mathbb{P}^2 = \mathbb{P}(V) \) (those that are a union of two lines or a “double line”) forms a proper projective subvariety of \( \mathbb{P}^5 = \mathbb{P}(\text{Sym}^2(V^*)) \) isomorphic to a certain projection of the Segre image of \( \mathbb{P}^2 \times \mathbb{P}^2 \) in \( \mathbb{P}^8 \) to \( \mathbb{P}^5 \). What is the dimension of the subvariety of degenerate conics?\(^2\)

   b). Show that the subset of “double lines” forms a proper closed subset of the space of all conics isomorphic to the Veronese surface in \( \mathbb{P}^5 \).

5. **Resultant.** Let \( F \) and \( G \) be two homogenous polynomials in \( k[U,V] \), of degrees \( m \) and \( n \) respectively. Let \( \text{Sym}^{n+m-1}(k^2)^* \) denote the vector space of homogeneous polynomials in \( k[U,V] \) of degree \( m+n-1 \).

   a). Show that \( F \) and \( G \) have a common factor if and only if the subvector spaces \( V_F \) and \( V_G \) of \( \text{Sym}^{n+m-1}(k^2)^* \) of polynomials divisible by \( F \) (respectively \( G \)) meet non-trivially.

   b). Show that \( F \) and \( G \) have a common factor if and only if the polynomials

   \[
   U^{n-1}F, U^{n-2}VF, \ldots, V^{n-1}F, U^{m-1}G, U^{m-2}VG, \ldots, V^{m-1}G
   \]

   are linearly dependent.

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\(^1\)Caution: the meaning of the ubiquitous phrase “general position” in algebraic geometry varies depending on the context, even on this very problem set!

\(^2\)Hint: \( \mathbb{P}^2 \times \mathbb{P}^2 \) is really \( \mathbb{P}(V^*) \times \mathbb{P}(V^*) \).
c). Show that \( F \) and \( G \) have a common factor if and only if the determinant of a certain \((m+n) \times (m+n)\) matrix formed from the coefficients of \( F \) and \( G \) is zero. This matrix is called the \emph{resultant} of \( F \) and \( G \).

d). Assume \( k = \bar{k} \). Let \( \mathbb{P}(\text{Sym}^m(k^2)^*) \) and \( \mathbb{P}(\text{Sym}^n(k^2)^*) \) be the projective spaces of all projective sets of points of degree \( m \) and \( n \) respectively.\(^3\) Consider the subset \( \Gamma \subset \mathbb{P}(\text{Sym}^m(k^2)^*) \times \mathbb{P}(\text{Sym}^n(k^2)^*) \) of pairs of point sets that have a point in common. Prove the subset \( \Gamma \) is a non-empty Zariski closed set.

e). Can you think of a way to generalize this problem to polynomials in more than two variables?

6. **Blowing up.** Let \( X \subset \mathbb{A}^2 \times \mathbb{P}^1 \) be the set of pairs \((p, \ell), \) where \( p \in \mathbb{A}^2 \) and \( \ell \) is a line through the origin in \( \mathbb{A}^2 \) containing \( p \).

a). Prove that \( X \) is a closed set in \( \mathbb{A}^2 \times \mathbb{P}^1 \). Find explicit defining equations.

b). Consider the natural map \( \pi : X \to \mathbb{A}^2 \) given by projection onto the first coordinate. Show that this map is a surjective regular map. For each point \( p \) of \( \mathbb{A}^2 \), describe the preimage set \( \{ \pi^{-1}(p) \} \) (in particular, what are its defining equations? what well-known variety is it isomorphic to? its dimension?). Is \( \pi \) finite?

c). Consider the natural map \( \eta : X \to \mathbb{P}^1 \) given by projection onto the second factor. Show that it is a surjective regular map. Describe the preimage of each point \( p \in \mathbb{P}^1 \). Is this map finite? This map defines what is called the \emph{tautological line bundle} on \( \mathbb{P}^1 \). Without going into technicalities about the definition of line bundles, why is this name justified?

7. **Diagonal Maps.** Consider the diagonal mapping \( \Delta : \mathbb{P}^n \subset \mathbb{P}^n \times \mathbb{P}^n \) sending each \( x \) to the pair \((x, x)\).

a). Prove that \( \Delta \) defines an isomorphism between \( \mathbb{P}^n \) and some closed set of \( \mathbb{P}^n \times \mathbb{P}^n \), and find an explicit set of bihomogeneous generators for the ideal of the image.

b). For any quasi-projective \( V \), show that the diagonal \( \Delta_V = \{(x, x)|x \in V\} \) is closed in \( V \times V \). [By definition, an abstract variety \( X \) is \emph{separated} if the diagonal in \( X \times X \) is closed.\(^4\) Separated is a property much like the Hausdorff property for topological spaces; in fact, from Math 591 you know that a topological space \( Y \) is Hausdorff if and only if the diagonal is closed in \( Y \times Y \). Does this mean all quasi projective varieties are Hausdorff? Explain!]

c). Prove that the intersection of two affine open subsets of a quasi-projective variety is affine. [By definition, a variety \( V \) is affine if it is isomorphic to an irreducible closed set in \( \mathbb{A}^n \).]

7. **A typical finite map.** Let \( V \) be an irreducible hypersurface in \( \mathbb{P}^n \) of degree \( d \) over an algebraically closed field of characteristic zero.

a). Show that every line in \( \mathbb{P}^n \) intersects \( V \) in exactly \( d \) points (counting multiplicity), unless it lies on \( V \). Part of the point here is to make sense of “counting multiplicity” of the intersection.

b). Pick any \( p \notin V \). Show the the projection from \( p \) to any hyperplane gives a surjective regular map \( V \to \mathbb{P}^{n-1} \). Explain why we expect this map to be “typically \( d \)-to-1.”

c). Show that we can choose coordinates so that \( p = [0:0: \ldots :0:1] \), \( \mathbb{P}^{n-1} = V(x_n) \), and the equation for \( V \) has the form \( x_n^d + a_1x_{n-1}^d + \ldots + a_d \), where the \( a_i \) are homogeneous of degree \( i \) in the variables \( x_0, \ldots, x_{n-1} \). Explicitly describe how to find the preimage of a point \( q \in \mathbb{P}^{n-1} \) in terms of these choices.

d). Prove that \( \pi \) is finite, directly from the definition of finiteness (as defined in Shafarevich.)

e). The points whose pre-image fails to have precisely \( d \) distinct points are called \emph{ramification points}. Prove that the set of ramification points (the ramification locus) is a proper Zariski closed subset of \( \mathbb{P}^{n-1} \), in fact, a hypersurface in \( \mathbb{P}^{n-1} \). (Hint: Remember a criterion from field theory for when a polynomial in a single variable has a repeated root.)

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\(^3\)A \emph{projective set of points of degree} \( n \) \emph{means} a \emph{hyper surface of degree} \( n \) in \( \mathbb{P}^1 \).

\(^4\)Many authors include separatedness as part of the definition of a variety; this rules out varieties like the “bug eyed line” where two copies of \( \mathbb{A}^1 \) are glued together to create a line with two origins.