Math 631: Problem Set 6

Due Monday October 21, 2013

1. Find the dimensions of the following varieties (now with proof, no appealing to intuition):
a). The affine cone over a projective variety of dimension $d$.
b). The variety of $m$ by $n$ matrices of rank $t$,
c). The twisted cubic in $\mathbb{P}^3$.
d). The Segre variety $\Sigma_{de}$ in $\mathbb{P}^{de+d+e}$.
e). Fix a point $Q$ and any other points $P_1,\ldots,P_t$ in $\mathbb{P}^n$. Let $\mathcal{X}$ be the subset of hypersurfaces of degree $d$ passing through $Q$ but none of the $P_i$. Show that $\mathcal{X}$ is a quasiprojective variety contained in $\mathbb{P}(\text{Sym}^d(V^*))$, and find its dimension.

2. More on Dimension. a). Show that if $V$ is an projective variety, and $V(F)$ is a hypersurface, then $V(F) \cap V$ has dimension exactly one less than dimension $V$, unless $V \subset V(F)$.
b). Define the dimension of a variety $V$ at a point $x$ to be the length of the longest chain of closed irreducible subvarieties of $V$ containing $x$. Denote this quantity by $\dim_x(V)$. Prove that $\dim_x(V)$ is the same for all points $x \in V$, and in this case, $\dim_x(V)$ agrees with the definition given in class (the transcendence degree of the function field over the ground field). [Hint: use 1e.]
c). For any quasi-projective variety $W$, explain why its dimension is equal to the Krull dimension of the coordinate ring of any dense open affine subset. Is the same true if $W$ is not irreducible?
d). Show that for any point $x$ on a variety $V$ of dimension $d$, there exist regular functions $f_1,\ldots,f_d$ in some neighborhood $U$ of $x$ such that, on $U$, the common zero set of the $f_i$ is precisely $x$. [In commutative algebra, these $f_i$ are called a system of parameters.] Can we assume the $f_i$ are (restrictions of) linear functions on an affine chart?

3. Degree of a Finite Map. Let $\phi : X \to Y$ be a finite morphism of irreducible varieties.
a). Explain how $\phi$ induces an identification of $k(Y)$ with a subfield of $k(X)$. We define the degree of $\phi$ to be the degree of this field extension.
b). Find the degree of the “projection from $p$ map” studied in problem 8 on homework set 5.
c). Compute the degree of $\mathbb{P}^1 \to \mathbb{P}^1$ sending $[s : t] \mapsto [s^d : t^d]$. Describe the fibers.
d). What is the degree of an isomorphism $X \to Y$?

\[1\text{It is also true, but substantially harder to show, that all chains of irreducible closed subvarieties of a quasi-projective variety have the same length.}\]
4. Another interesting finite map. Fix a projective variety \( V \subset \mathbb{P}^n \). Let \( F_0, \ldots, F_m \) be homogeneous polynomials of degree \( d \) in \( k[x_0, \ldots, x_n] \) that do not simultaneously vanish at any point of \( V \). Consider the map \( \phi : V \to \mathbb{P}^m \) sending a point \( x \) to \([F_0(x) : \ldots : F_m(x)]\).

a). Prove that \( \phi \) is finite onto its image by reducing the case where \( d = 1 \) and using the fact that projections from linear spaces are finite.

b). Prove \( \phi \) is finite onto its image by directly thinking about the preimage of each point and applying the theorem (see Shafarevich) that for projective varieties, a dominant morphism is finite if and only if it has finite fibers.

c). In the case \( V = \mathbb{P}^n \), can you give any bounds on the number of points in the pre-image of a typical point?

5. Local ring at a sub variety. Fix a quasi-projective variety \( V \), and a closed sub variety \( W \). Define the local ring of \( V \) along \( W \) to be the following:

\[
\mathcal{O}_{W,V} := \{ \phi \in k(V) \mid \phi \in \mathcal{O}_V(U) \text{ for some open set } U \text{ such that } U \cap W \neq \emptyset \}.
\]

a). Confirm that if \( W \) is a point, this agrees with the definition already given for the local ring at a point on a prior assignment. Check also that \( \mathcal{O}_{W,V} \) is actually a local ring.

b). Show that if \( V \) is affine, \( \mathcal{O}_{W,V} \) is the localization of \( k[V] \) at the prime ideal \( \mathcal{I}(W) \). What can be said in the non-affine case?

c). Compute the local ring of \( \mathbb{P}^2 \) at the point \([1 : 0 : 0]\), and at the line given by \( \mathcal{V}(x_0) \). Express your answer so elements of the ring are actually functions on (open sets of) \( \mathbb{P}^2 \).

6. The Plücker Embedding. a). Let \( V \) be an \( n \)-dimensional vector space, For \( d \leq n \), let \( N = \binom{n}{d} \). Show that there is a well defined map

\[
\phi : \mathcal{G}_d(V) \to \mathbb{P}^{N-1} = \mathbb{P}(\wedge^d V)
\]

which sends any \( d \)-dimensional subspace \( W \) to \( \wedge^d W \subset \wedge^d V \). Show also that \( \phi \) is one-to-one onto its image, and that the image is precisely the set of (one dimensional) subspaces of \( \wedge^d V \) spanned by indecomposable vectors in \( \mathbb{P}(\wedge^d V) \) (that is, vectors of the form \( v_1 \wedge v_2 \ldots \wedge v_d \)).

b). Show that \( \phi \) can be expressed explicitly in coordinates as taking a \( d \times n \) matrix representing a \( d \)-dimensional space \( W \) to an \( N \)-tuple consisting of its maximal minors.

c). Show that a point \([\omega]\) in \( \mathbb{P}(\wedge^d(V)) \) (represented by a vector \( \omega \in \wedge^d(V) \)) lies in the image of \( \phi \) if and only if the map

\[
\wedge \omega : V \to \wedge^{d+1} V
\]

sending \( v \) to \( v \wedge \omega \) is rank \( n - d \). (Hint: First show that a vector \( \omega \in \wedge^d V \) is indecomposable if only if the space of vectors that “divide” it in the exterior algebra has dimension \( d \).)

d.) Show that \( \phi \) identifies \( \mathcal{G}_d(V) \) with a Zariski closed subset of \( \mathbb{P}(\wedge^d V) \). Thus Grassmannians are projective varieties!

e). Recall that on Problem Set 4, you found a nice affine cover of the Grassmannian. Show that this cover corresponds to the standard affine open cover under the identification of \( \mathcal{G}_d(V) \) with a projective variety in \( \mathbb{P}(\wedge^d V) \) via the embedding \( \phi \). Is \( \mathcal{G}_d(V) \) irreducible? What is its dimension?

\[2\]If you used (c) to prove (d), the equations you get do not generate the full ideal of all homogeneous polynomials equations vanishing on \( \mathcal{G}_d(V) \). It turns out that this ideal is generated by quadratic polynomials, called Plücker relations, described on p 42 of Shafarevich.