Math 631: Problem Set 5

Due Friday October 12, 2012

1. Closed sets of $\mathbb{P}^n \times \mathbb{P}^m$. Recall that the closed sets in $\mathbb{P}^n \times \mathbb{P}^m$ are given by bihomogeneous polynomials (see Shaf 5.1).

a). Let $X$ be a hyperplane section of the Segre variety $\Sigma_{nm}$ in $\mathbb{P}^{(m+1)(n+1)-1}$, that is, the intersection of a hyperplane in $\mathbb{P}^{(m+1)(n+1)-1}$ with the image of $\mathbb{P}^n \times \mathbb{P}^m$ under the Segre map $\sigma_{nm}$. Explicitly describe the bihomogeneous polynomials that define the corresponding closed subset of $\mathbb{P}^n \times \mathbb{P}^m$.

b). Let $X = \mathbb{V}(x_0^2 x_1^2 y_0 y_1 + x_0^2 y_1^2 + x_0^4 x_1 y_0^2)$ be a closed set in $\mathbb{P}^1 \times \mathbb{P}^1$ (where homogeneous coordinates are $x_0 : x_1$ on first copy and $y_0 : y_1$ on second copy of $\mathbb{P}^1$). Find defining equations for the image of $X$ under the (restriction of) the segre embedding $\sigma_{1,1}$ in $\mathbb{P}^3$, with coordinates $z_{00}, z_{01}, z_{10}, z_{11}$.

2. Resultant. Let $F$ and $G$ be two homogeneous polynomials in $k[U, V]$, of degrees $m$ and $n$ respectively. Let $\text{Sym}^{n+m-1}(k^2)^*$ denote the vector space of homogeneous polynomials in $k[U, V]$ of degree $m + n - 1$.

a). Show that $F$ and $G$ have a common factor if and only if the subvector spaces $V_F$ and $V_G$ of $\text{Sym}^{n+m-1}(k^2)^*$ of polynomials divisible by $F$ (respectively $G$) meet non-trivially.

b). Show that $F$ and $G$ have a common factor if and only if the polynomials

$$U^{n-1}F, U^{n-2}VF, \ldots, V^{n-1}F, U^{m-1}G, U^{m-2}VG, \ldots, V^{m-1}G$$

are linearly dependent.

c). Show that $F$ and $G$ have a common factor if and only if the determinant of a certain $(m+n) \times (m+n)$ matrix formed from the coefficients of $F$ and $G$ is zero. This matrix is called the \textit{resultant} of $F$ and $G$.

d). Assume $k = \bar{k}$. Let $\mathbb{P}(\text{Sym}^m(k^2)^*)$ and $\mathbb{P}(\text{Sym}^n(k^2)^*)$ be the projective spaces of all homogeneous polynomials in $U$ and $V$ of degree $m$ and $n$ respectively. Consider the subset $\Gamma = \{(F, G) \mid \mathbb{V}(F) \cap \mathbb{V}(G) \neq \emptyset \text{ in } \mathbb{P}^1\}$ of $\mathbb{P}(\text{Sym}^m(k^2)^*) \times \mathbb{P}(\text{Sym}^n(k^2)^*)$. Prove the subset $\Gamma$ is a non-empty Zariski closed set.

3. Blowing up. Let $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ be the set of pairs $(p, \ell)$, where $p \in \mathbb{A}^2$ and $\ell$ is a line through the origin in $\mathbb{A}^2$ containing $p$.

a). Prove that $X$ is a closed set in $\mathbb{A}^2 \times \mathbb{P}^1$. Find explicit defining equations.

b). Consider the natural map $\pi : X \rightarrow \mathbb{A}^2$ given by projection onto the first coordinate. Show that this map is a surjective regular map. For each point $p$ of $\mathbb{A}^2$, describe the preimage set $\{\pi^{-1}(p)\}$ (in particular, what are its defining equations? what well-known variety is it isomorphic to? its dimension?). Is $\pi$ finite?

c). Consider the natural map $\eta : X \rightarrow \mathbb{P}^1$ given by projection onto the second factor. Show that it is a surjective regular map. Describe the preimage of each point $p \in \mathbb{P}^1$. Is this map finite? This map defines what is called the \textit{tautological line bundle} on $\mathbb{P}^1$. Without going into technicalities about the definition of line bundles, why is this name justified?
4. **Family of Degenerate Conics.** For this problem assume the field does not have characteristic 2.

a). Show that the subset of degenerate conics in $\mathbb{P}^2 = \mathbb{P}(V)$ (those that are a union of two lines or a “double line”) forms a proper projective subvariety of $\mathbb{P}^5 = \mathbb{P}(\text{Sym}^2(V^*))$ isomorphic to a certain projection of the Segre image of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ to $\mathbb{P}^5$. What is the dimension of the subvariety of degenerate conics?\(^1\)

b). Show that the subset of “double lines” forms a proper closed subset of the space of all conics isomorphic to the Veronese surface in $\mathbb{P}^5$.

5. **A typical finite map.** Let $V$ be an irreducible hypersurface in $\mathbb{P}^n$ of degree $d$ over an algebraically closed field of characteristic zero.

a). Show that every line in $\mathbb{P}^n$ intersects $V$ in exactly $d$ points (counting multiplicity), unless it lies on $V$. Part of the point here is to make sense of “counting multiplicity” of the intersection.

b). Pick any $p \notin V$. Show the the projection from $p$ to any hyperplane gives a surjective regular map $V \to \mathbb{P}^{n-1}$. Explain why we expect this map to be “typically $d$-to-$1$.”

c). Show that we can choose coordinates so that $p = [0 : 0 : \ldots : 0 : 1]$, $\mathbb{P}^{n-1} = V(x_n)$, and the equation for $V$ has the form $x_0^d + a_1x_0^{d-1} + \ldots + a_d$, where the $a_i$ are homogeneous of degree $i$ in the variables $x_0, \ldots, x_{n-1}$. Explicitly describe how to find the preimage of a point $q \in \mathbb{P}^{n-1}$ in terms of these choices.

d). Prove that $\pi$ is finite, directly from the definition of finiteness (as defined in Shafarevich.)

e). The points whose pre-image fails to have precisely $d$ distinct points are called ramification points. Prove that the set of ramification points (the ramification locus) is a proper Zariski closed subset of $\mathbb{P}^{n-1}$, in fact, a hypersurface in $\mathbb{P}^{n-1}$. (Hint: Remember a criterion from field theory for when a polynomial in a single variable has a repeated root.)

6. **Every projective variety is defined by Quadrics.** Let $X$ be an arbitrary projective variety in $\mathbb{P}^n$.

a). Show that $X$ can be described as the common zero set of a collection of homogeneous polynomials having all the same degree.

b). Show that $X$ is isomorphic to a “linear section” of a Veronese $n$-fold. That is, there is some $d$ such that $X$ is isomorphic to a variety of the form $V_d \cap L$ where $L$ is a linear variety in $\mathbb{P}^N$ and $V_d$ is the image of $\mathbb{P}^{n}$ under the Veronese map $\nu_d$.

c). Show that $X$ is isomorphic to an intersection of quadrics. (A quadric is a hypersurface in projective space defined by a homogeneous polynomial of degree two.)

7. **Diagonal Maps.** Consider the diagonal mapping $\Delta : \mathbb{P}^n \subset \mathbb{P}^n \times \mathbb{P}^n$ sending each $x$ to the pair $(x, x)$.

a). Prove that $\Delta$ defines an isomorphism between $\mathbb{P}^n$ and some closed set of $\mathbb{P}^n \times \mathbb{P}^n$, and find an explicit set of bihomogeneous generators for the image.

b). For any quasiprojective $V$, show that the diagonal $\Delta_V = \{(x, x) | x \in V\}$ is closed in $V \times V$.

c). Prove that the intersection of two affine open subsets of a quasi-projective variety is affine. (By definition, a quasi-projective variety $V$ is affine if it is isomorphic as a quasi-projective variety to a closed set in $\mathbb{A}^n$; that is, there are mutually inverse regular maps between $V$ and a closed set in $\mathbb{A}^n$.)\(^2\)

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\(^1\)Hint: $\mathbb{P}^2 \times \mathbb{P}^2$ is really $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$.

\(^2\)This is one property of quasi-projective varieties that can fail for abstract varieties as we have defined them. Usually, one adds the property that the diagonal is closed (such a variety is said to be "separated") to the definition of an abstract variety to rule out such pathological behavior. For example, as we defined abstract variety, one can put the structure of an abstract variety on the affine line with a "doubled origin", a slight variant on the classic example of a non-Hausdorff space from Math 591. Indeed, separatedness for algebraic varieties is the analog of "Hausdorff" in topology.