Characteristic $p$ Techniques in Commutative Algebra and Algebraic Geometry
Math 732 - Winter 2019

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The goal of this course is to introduce the Frobenius morphism and its uses in commutative algebra and algebraic geometry.

These characteristic $p$ techniques have been used in commutative algebra, for example, to establish that certain rings are Cohen-Macaulay, as in the famous Hochster-Roberts theorem for rings of invariants (over fields of arbitrary characteristic). In more geometric settings, characteristic $p$ techniques can help analyze or quantify how singular—that is, how far from being smooth—a particular variety may be, or to establish that the singularities are suitably mild.

Vast bodies of research in birational algebraic geometry that had been developed for complex varieties using differential forms and $L^2$-techniques have now been seen to have “characteristic $p$” analogs. These can be used to recover much of the theory in the complex setting, but even better, can serve as replacements to the analytic techniques to establish results for varieties of prime characteristic $p$. Examples of such analogs we will discuss in this course include the F-pure threshold and test ideals, which are “characteristic $p$ analogs” of the log canonical threshold and multiplier ideals in complex geometry.

In this course, we will introduce this machinery, while also introducing and/or reviewing many of the sub-areas of commutative algebra and algebraic geometry where it has been useful. We begin with Kunz’s characterization of regularity (smoothness) as the flatness of Frobenius. First, of course, we review regular local rings more broadly, including Serre’s famous characterization in terms of global dimension. From there, we will consider classes of “mild singularities” for complex varieties, reviewing the definitions of rational, log canonical and log terminal singularities which are defined using resolutions of singularities. These all have “characteristic $p$ analogs” defined using Frobenius, which we will explore in depth. We will then study some global implications of Frobenius splitting, including vanishing theorems for line bundles and characterizations of Fano (or nearly Fano) varieties. As time and student interest permit, we may study singularity invariants such as the log canonical threshold and its “characteristic $p$ analog” called the F-pure threshold, as well as multiplier and test ideals, Hilbert-Kunz Multiplicity and/or F-signature.
CHAPTER 1

Introduction

1. Overview

Let $X$ be a Noetherian scheme of prime characteristic $p$. The **Frobenius** morphism is the scheme map

$$F : X \to X \quad \mathcal{O}_X \to F_*\mathcal{O}_X$$

which is the *identity map* on the underlying topological space $X$ but the $p^{th}$ power map on sections. For example, when $X = \text{Spec } R$ is affine, the Frobenius map is essentially just the ring homomorphism $R \to R$ sending $r \mapsto r^p$.

An important special case to keep in mind is the case where $X$ is a variety over an algebraically closed field $K$ of characteristic $p > 0$.

The Frobenius map is a powerful tool, both in commutative algebra and algebraic geometry. Some of its uses include

(a) Detecting regularity of $X$: A famous theorem of Kunz says that $X$ is regular if and only if the Frobenius morphism is flat. For varieties, this says that $X$ is smooth if and only if $F_*\mathcal{O}_X$ is a locally free sheaf (or vector bundle).

(b) By relaxing the flatness condition for the Frobenius in various ways, we get a host of other “mild singularity” classes in prime characteristic, with names you may have heard before such as F-regular and F-pure. Many theorems that hold for smooth varieties over fields of characteristic $p$ can be extended to these classes of “F-singularities.”

(c) Frobenius can be used to detect “mild” singularities of complex varieties as well. Singularities of importance in complex birational geometry such as Kawamata log terminal or canonical singularities can be understood by “reduction to prime characteristic,” where surprisingly checkable criteria for these singularities can be given in terms of the Frobenius map.

(d) In commutative algebra, the Frobenius can be used to proving certain rings are Cohen-Macaulay or have other nice properties. Again, by reduction to prime characteristic, such techniques can be applied to any ring containing a field.

(e) The Frobenius map can be used to prove vanishing theorems for cohomology of line bundles on varieties of prime characteristic under certain conditions. Thus Frobenius can often be used when resolution of singularities, Kodaira Vanishing, or other tools fail (or are unknown) in characteristic $p$. 

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(f) Frobenius can be used to define numerical invariants of singularities. For example, the **F-pure threshold** is an invariant defined using the Frobenius which is an analog of the analytic index of singularities (or log canonical threshold) for complex varieties. A smooth point on a hypersurface always has F-pure threshold 1, but singular points have smaller (positive) thresholds—the smaller, the more singular.

As one concrete example, consider the simple cusp defined by \( y^2 = x^3 \). This singularity has F-pure threshold equal to

\[
\begin{align*}
&\frac{1}{2} \quad \text{in char } 2 \\
&\frac{2}{3} \quad \text{in char } 3 \\
&\frac{5}{6} - \frac{1}{6p} \quad \text{if char } p = 5 \text{ mod } 6 \\
&\frac{5}{6} \quad \text{if char } p = 1 \text{ mod } 6.
\end{align*}
\]

This reflects the fact that the cusp is somehow most singular in characteristic 2, slightly less singular in characteristic 3, and increasing less singular for large \( p \). For infinitely many \( p \) (namely, those \( p \) congruent to 5 mod 6), it is not really anymore singular than it is over complex numbers, in the sense that the **log canonical threshold** of the cusp over \( \mathbb{C} \) is also \( \frac{5}{6} \). The fact that the cusp is “more singular” in some characteristics, even infinitely many, than it is over \( \mathbb{C} \) is deeply connected to arithmetic issues related to supersingularity for elliptic curves. Here is one specific long-standing **conjecture**: for a given polynomial with integer coefficients (say), the log canonical threshold of the corresponding hypersurface over \( \mathbb{C} \) is equal to the F-pure threshold of the hypersurface considered over \( \mathbb{F}_p \) for infinitely many \( p \).

Our goal will be to expand on all of these topics over the semester.

1.0.1. **Local Properties.** Let \( \mathcal{P} \) be a property of schemes (or, say, of varieties or of topological spaces). We say that “\( \mathcal{P} \) is a local property” if for any scheme \( X \), \( X \) has property \( \mathcal{P} \) if and only if for all \( x \in X \), there is an open neighborhood \( U \) where property \( \mathcal{P} \) holds. Note the locus \( X_0 \) of points of an arbitrary scheme \( X \) where the local property \( \mathcal{P} \) holds will be an open set of \( X \). For example, smoothness and normality are both local properties of complex varieties. Any property characterizing the ‘singularities’ of a scheme ought to be a local property.

Typically, a property of a scheme \( X \) may be defined in terms of ring properties for its stalks \( \mathcal{O}_{X,x} \). That is, we may say that a point \( x \) has property \( \mathcal{P} \) if and only if the stalk \( \mathcal{O}_{X,x} \) has property \( \mathcal{P} \). Reducedness and normality are such a properties of schemes, as are more exotic properties such as Cohen-Macaulayness and Gorenstein-ness. Property \( \mathcal{P} \) holding for \( X \) if and only if it holds for all points \( x \in X \) follows immediately if \( \mathcal{P} \) is a local property, but this is weaker assumption in general.

Note that if \( \mathcal{P} \) is such a local property and some stalk \( \mathcal{O}_{X,x} \) has property \( \mathcal{P} \), then all further localizations \( \mathcal{O}_{X,y} \) must have property \( \mathcal{P} \). Indeed, to say that \( \mathcal{O}_{X,y} \) is a further localization of \( \mathcal{O}_{X,x} \) is to say that the point \( x \) is in the closure of \( y \), so every open set containing \( x \) will also contain \( y \).
Non-local properties include connectedness, or the property that the sheaf \( \omega_X \) has a non-zero section.

It is not always obvious whether or not a given property is local, even when we expect it to be. For example, the property of \textbf{regularity} for Noetherian schemes, introduced mid-last century as a scheme analog of smoothness for algebraic varieties, was long expected to be a local property. However, until the advent of homological algebra, it wasn’t even known that a localization of a regular local ring is regular; this was settled in the 1950’s by Serre (and independently Auslander-Buchsbaum) who gave new homological characterizations of regularity. Unfortunately, it turns out that regularity is not a local property \cite{Hoc}, though it is for large classes of rings with mild hypothesis.
CHAPTER 2

Regularity and Kunz’s theorem

1. Frobenius

We begin with the “local” picture, studying Frobenius for rings.

Setting 1.1. The word “ring” in these notes always means a commutative ring with unity (unless explicitly stated otherwise). Our rings are usually Noetherian. Recall that a Noetherian ring is called local if it has a unique maximal ideal. The notation \((R, m, K)\) will always denote a Noetherian local ring with maximal ideal \(m\) and residue field \(K = R/m\).

We start with some examples of the types of rings we are interested in.

Example 1.2. (a) The polynomial ring \(\mathbb{C}[x_1, \ldots, x_n]\); this is the ring of polynomial (or regular) functions on the variety \(\mathbb{C}^n\).
(b) The polynomial ring \(K[x_1, \ldots, x_n]\). When \(K\) is algebraically closed, we can view this as the ring of regular functions on the variety \(K^n\).
(c) The ring \(\mathcal{O}_V(U)\) of regular functions on some open set \(U\) of a variety \(V\) over some field \(K\).
(d) A finitely generated \(K\) algebra \(R = K[x_1, \ldots, x_n]/I\) where \(I\) is some ideal \(K[x_1, \ldots, x_n]\). Every ring of the form (c) is of this form, provided \(U\) is an affine open set.
(e) More explicitly, if \(V \subset K^n\) is an affine algebraic variety over an algebraically closed field \(K\) (with fixed closed embedding into \(K^n\)), then
\[
\mathcal{O}_V(V) \cong K[x_1, \ldots, x_n]/I_V
\]
where \(I_V\) is the radical ideal of all polynomial functions vanishing on \(V\). The isomorphism is given by the restriction map \(K[x_1, \ldots, x_n] \to \mathcal{O}_V(V)\) sending \(\phi \mapsto \phi|_V\). This follows from Noether’s First Isomorphism Theorem, since the restriction map is surjective with kernel is \(I_V\).
(f) A localization of any of the rings \(R\) above at any prime ideal \(P\). This is a local ring \(R_P\), whose maximal ideal is \(PR_P\) whose residue field \(R_P/PR_P\) can be denoted \(k(P)\).
(g) The local ring (or stalk) \(\mathcal{O}_{V,p}\) of a point \(p\) on a variety \(V\). Explicitly, if \(V\) is as in (e), this ring is
\[
\frac{K[x_1, \ldots, x_n]/m_p}{I_V}
\]
where $m_p$ is the maximal ideal of all regular functions vanishing at $p$. We can also take the stalk at non-closed points, which amounts to localizing at a non-maximal prime ideal of $\frac{K[x_1,\ldots,x_n]}{I_V}$. We can also take the stalk at non-closed points, which amounts to localizing at a non-maximal prime ideal of $K[x_1,\ldots,x_n]$. This can be viewed as the completion of the polynomial ring $K[x_1,\ldots,x_n]$ at the maximal ideal $(x_1,\ldots,x_n)$.

(h) The local ring $\mathcal{O}_{X,p}$ of a scheme $X$ at a point $p$.

(i) The power series ring $K[[x_1,\ldots,x_n]]$. This can be viewed as the completion of the polynomial ring $K[x_1,\ldots,x_n]$ at the maximal ideal $(x_1,\ldots,x_n)$.

(j) Quotients $k[[x_1,\ldots,x_n]]/I$ of power series rings. These can also be viewed as completions of local rings $\mathcal{O}_{V,P}$ at points, in which case the field $k$ is (non-canonically) identified with the residue field $k(P)$ of the point. For example, $\mathbb{C}[[x,y]]/(y^2 - x^3)$ is the completion of the local ring of cuspidal singularity $y^2 = x^3$ over $\mathbb{C}$.

Remark 1.3. All the examples above are Noetherian and contain a field. This is the main setting in which we will be able to use Frobenius to prove theorems. Of course, most of the action takes place when that field has prime characteristic $p > 0$. For applications, we can use “reduction to characteristic $p > 0$" to prove statements in the case where the field is characteristic zero. This will be discussed in detail later.

Rings that do not contain a field, such as $\mathbb{Z}$, its localization $\mathbb{Z}_p$, the $p$-adic integers, or many algebras over any of these, are said to be of mixed characteristic. We will not be considering such rings in this course. Recent progress due to Peter Scholze, Bhargav Bhatt, and others now allows some of the “characteristic p techniques” discussed here to be extended to mixed characteristic. If you are interested in this case, I suggest you look up courses given by Bhatt on the theory of perfectoid rings.

Let’s now work in positive prime characteristic $p$. The following Key Lemma is the heart of it all.

Lemma 1.4. Let $R$ be a commutative ring of prime characteristic $p$. The map $F : R \to R$ which sends $r \mapsto r^p$ is a ring homomorphism.

Proof. Since $R$ is commutative, $F(rr') = (rr')^p = r^p r'^p = F(r)F(r')$. The additive part is trickier. First,

$$F(r + r') = (r + r')^p = r^p + \binom{p}{1} r^{p-1} r' + \ldots + \binom{p}{p-1} r'r'^{p-1} + r'^p.$$  

Now it is an easy exercise to check that $p$ divides the binomial coefficients $\binom{p}{i}$ for all $0 < i < p$. So the “mixed terms” all vanish when $R$ has characteristic $p$, and $F(r + r') = r^p + r'^p = F(r) + F(r')$. □

This $p$-th power map is called the Frobenius morphism of the ring $R$. It turns out to be a very useful tool in prime characteristic commutative algebra (and algebraic geometry). Already, we can see that the simplest kind of “singularity” can be detected with Frobenius:

Lemma 1.5. Let $R$ be a ring of prime characteristic $p$. Then Frobenius is injective if and only if $R$ is reduced (has no non-zero nilpotents).
Proof. Suppose first that Frobenius is injective. Take some nilpotent \( x \in R \). So
\[ x^n = 0 \in R \]
for some \( n \), and we need to show that \( x = 0 \). Since \( x^n = 0 \), we know that \( x^{pe} = 0 \) for any integer \( e > 0 \) for which \( pe \geq n \). But then \( F^e = F \circ F \circ \cdots \circ F \) sends \( x \mapsto x^{pe} = 0 \). Since Frobenius (and hence its iterates) is injective, we conclude that \( x = 0 \), as desired.

Conversely, if \( R \) is reduced, \( F \) is obviously injective because \( F(x) = x^p = 0 \) implies that \( x = 0 \). QED.

Remark 1.6. The Frobenius morphism is almost never surjective in a Noetherian ring. An exception is the ring \( \mathbb{F}_p \).

On the other hand, the Frobenius morphism is the identity map on the topological space \( \text{Spec} R \). Recall that for any ring map \( \phi : R \to S \), there is an induced map
\[ \text{Spec} S \to \text{Spec} R \quad P \mapsto \phi^{-1}(P) \]
that is continuous in the Zariski topology.

Lemma 1.7. Let \( R \) be a ring of prime characteristic \( p \). The map of affine schemes induced by the Frobenius map of \( R \)
\[ \text{Spec} R \to \text{Spec} R \quad P \mapsto F^{-1}(P) \]
is the identity map.

Proof. To see that \( F^{-1}(P) \subseteq P \), take any \( x \in F^{-1}(P) \). This means that \( x^p \in P \), and since \( P \) is prime, also \( x \in P \).

For the other direction, take any \( x \in P \). Then also \( x^p \in P \), so \( F(x) \in P \) and \( x \in F^{-1}(P) \). QED.

Combining these two easy lemmas, we can already say that the Frobenius morphism of \( \text{Spec} R \), while doing nothing to the underlying topological space, is a highly non-trivial morphism of schemes. Indeed, we just saw that it can tell us whether or not our scheme is reduced, the most basic restriction on its singularities.

An easy, but crucial, point is that Frobenius commutes with localization:

Lemma 1.8. Let \( R \) be a ring of prime characteristic \( p \). Let \( W \subset R \) be any multiplicative system. Then the diagram
\[
\begin{array}{ccc}
R & \xrightarrow{F} & R \\
\downarrow & & \downarrow \\
W^{-1}R & \xrightarrow{F} & W^{-1}R \\
\end{array}
\]
where the downward arrows are the canonical localization maps and the rightward arrows are the Frobenius maps on their respective rings, commutes.

This lemma, together with the fact that the Frobenius map induces the identity map on Spectra immediately implies that there is a well defined Frobenius morphism for any scheme of characteristic \( p \). By definition, a scheme \( (X, \mathcal{O}_X) \) has prime characteristic \( p \) if for every non-empty open set \( U \subset X \), the ring \( \mathcal{O}_X(U) \) has characteristic \( p \).
We can thus define the Frobenius morphism to be the identity map on the underlying topological space of $X$ and the $p$-th power map for each ring $\mathcal{O}_X(U)$. Because Frobenius commutes with localization, the Frobenius maps on an affine cover $\{\text{Spec } R_\lambda\}_{\lambda \in \Lambda}$ can be glued together to a produce the Frobenius morphism of $X$:

$$F : X \to X \quad \mathcal{O}_X \to F_\ast \mathcal{O}_X$$

given by the $p$-th power map on every local chart.

### 1.1. The Module defined by Frobenius and the functor $F_\ast$.
Whenever we have a ring homomorphism $f : R \to S$, we can view $S$ as an $R$-module via restriction of scalars: by definition, $r \in R$ acts on $s \in S$ by $r \cdot s = f(r)s$.

This is true in particular for the Frobenius map. Because the source and target of Frobenius are both the same ring $R$, this can be confusing. We therefore use the notation $F_\ast R$ to denote the target copy of $R$. As an $R$-module, the notation $F_\ast R$ denotes the abelian group $R$, but with $R$-action via Frobenius: so $r \in R$ acts on $s \in F_\ast R$ by $r \cdot s = r^p s$. With this convention, the Frobenius map

$$F : R \to F_\ast R \quad r \mapsto r^p$$

is not only a ring map, but also an $R$-module map. This notation is consistent with the standard notation for schemes: in general, a map of schemes $g : Y \to Z$ comes with a map of sheaves of rings $\mathcal{O}_Z \to g_\ast \mathcal{O}_Y$ on $Z$ which makes $g_\ast \mathcal{O}_Y$ into a module over $\mathcal{O}_Z$ (meaning, a sheaf of modules over the sheaf of rings $\mathcal{O}_Z$).

More generally $F_\ast$ is a functor from $R$-modules to $R$-modules (or quasi-coherent sheaves on $X$). This is simply the restriction of scalars functor which we can apply to any $R$-module. Indeed, if $M$ is an $R$-module then $F_\ast M$ is the $R$-module which is the same as $M$ as an abelian group but such that if $r \in R$, and $m \in F_\ast M$, then $r \cdot m = r^p m$. Because it can be confusing to remember which module $m$ is in, sometimes we write $F_\ast m$ instead of $m$. With this notation, $r \cdot F_\ast m = F_\ast r^p m$.

The functor $F_\ast$ is obviously exact, since it literally does nothing to the underlying abelian groups (or mappings) when we apply it; it only changes the way elements of $R$ act. For example, if $M \to N$ is a homomorphism of $R$-modules with kernel $A$ and cokernel $B$, then $F_\ast M \to F_\ast N$ is a map with kernel $F_\ast A$ and cokernel $F_\ast B$. For example, for a ring $R$ with maximal ideal $m$ and residue field $K$, the ring $F_\ast R$ has maximal ideal $F_\ast m$ and residue field $F_\ast K$, which is a degree $[K : K^p]$ extension of $K$.

Sometimes it is convenient to use a different notation for Frobenius, and indeed, most of the commutative algebra papers on the subject do. Let $R$ be a reduced ring and let $R^p$ denote the subring of $p$th powers of $R$. Then the map $R \to R^p$ which sends $r \to r^p$ is a ring isomorphism. The Frobenius map $F : R \to R$, which has image $R^p$, factors through the inclusion $R^p \hookrightarrow R$, and in fact can be identified with this inclusion.

Alternatively, we can let $R^{1/p}$ denote the ring of $p$th roots of all elements of $R$ (say, inside an algebraic closure of the fraction field of $R$). Again $R^{1/p}$ is a ring abstractly isomorphic to $R$ via the map $R^{1/p} \to R$ which sends $x \mapsto x^p$. In particular
the Frobenius on $R^{1/p}$ has image $R$ (inside $R^{1/p}$). Hence $F$ can also be viewed as the inclusion $R \subseteq R^{1/p}$. These two notations are essentially the same: we can either write $R^p$ for the source copy of $R$, and view the module as the $R^p$-module $R$, or equivalently, by taking $p$-th roots of everything, we can write $R^{1/p}$ for the target copy of $R$ and view the module as the $R$-module $R^{1/p}$. In different contexts, one or the other of these points of view may be more convenient.

In fact, the notation $R^{1/p}$ for the $R$-module $F_*R$ can be used even when $R$ is not reduced, provided we interpret $r^{1/p}$ as simply another way to write $F_*r$. Likewise, some authors use $m^{1/p} \in M^{1/p}$ to denote an element $F_*m \in F_*M$. This makes sense literally as a $p$-th root in many settings; for example, if $R$ is a domain and $I$ is an ideal, then we can write $I^{1/p}$ to denote the $p$-th roots of elements of $I$. Note that $I^{1/p}$ is an ideal of $R^{1/p}$, and in fact, the image of $I$ under the identification $R \leftrightarrow R^{1/p}$ sending $r \mapsto r^{1/p}$. Similarly, if $M$ is an $R$-submodule of the fraction field $K$ of $R$, then $M^{1/p}$ makes sense as a subset of $K^{1/p}$ (which in turn, is makes sense inside an algebraic closure of $K$).

Of course, this discussion applies just as well to the iterated Frobenius maps $F^e$. The $R$-module $F^e_*R$ is defined by restriction of scalars for the $e$-times iterated Frobenius map

$$R \to F^e_*R \quad r \mapsto r^{p^e},$$

which is both a ring map and an $R$-module map. We can identify $F^e_R$ with $R^{1/p^e}$.

Each of the different notations for the module $F_*R$ has advantages and disadvantages and each takes getting used to. We will switch between these descriptions freely.

**Example 1.9** (Polynomial ring in one variable). Consider $R = \mathbb{F}_p[x]$. Then it is easy to see that $R$ is a free $R^p$-module of rank $p$ with basis $1, x, \ldots, x^{p-1}$. Equivalently, $R^{1/p}$ is a free $R$-module with basis $1, x^{1/p}, \ldots, x^{(p-1)/p}$. In our preferred notation, which granted takes some getting used to, we see that $F_*R$ is a free $R$-module with basis $1, x, \ldots, x^{p-1}$. To avoid confusion, we can denote this basis by $F_*1, F_*x, \ldots, F_*x^{p-1}$ even though $F_*$, as a functor, doesn’t act on elements exactly.

**Example 1.10** (Polynomial ring in $n$ variables). Consider $R = \mathbb{F}_p[x_1, \ldots, x_n]$. Then $R$ is a free $R^p$-module of rank $p^n$ with basis $\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq p-1\}$. Equivalently, $R^{1/p}$ is a free $R$-module with basis $\{x_1^{a_1/p} \cdots x_n^{a_n/p} \mid 0 \leq a_i \leq p-1\}$. Similarly, $F_*R$ is a free $R$-module on the basis $\{x_1^{a_1} \cdots x_n^{a_n} \mid 0 \leq a_i \leq p-1\}$. This same basis works when $\mathbb{F}_p$ is replaced by any perfect field.

Often we will iterate Frobenius: the map $F^e : R \to R$ is the ring homomorphism sending $r \mapsto r^{p^e}$. Similarly, the notation $F^e_*R$ denotes the $R$-module which is $R$ as an abelian group but whose action by elements of $R$ is via $F^e$.

**Exercise 1.1.** Let $K$ be any field of characteristic $p$. Find a minimal generating set for $F^e_*K[x_1, \ldots, x_n]$ over $K[x_1, \ldots, x_n]$, and prove that it is a free basis. If $[K : K^p] = d < \infty$, prove that the rank of $F^e_*K[x_1, \ldots, x_n]$ over $K[x_1, \ldots, x_n]$ is $dp^{ne}$.

The situation is more complicated for non-polynomial rings.
Example 1.11. Consider $R = \mathbb{F}_p[a, b]/(a^3 - b^2) = \mathbb{F}_p[x^2, x^3] \subseteq \mathbb{F}_p[x]$. Let’s try to understand the structure of $R^{1/p}$ as an $R$-module.

Observe that $R$ has an $\mathbb{F}_p$-vector space basis consisting of all monomials $x^n$ where $n$ is any positive integer except 1. Thus $R^{1/p}$ has an $\mathbb{F}_p$-vector space basis consisting of all monomials $x^{n/p}$ where $n$ is any positive integer except 1, as well. Note that $R$ is $\mathbb{N}$-graded in a natural way, and that this grading is compatible with a natural $\frac{1}{p}\mathbb{N}$-grading on $R^{1/p}$.

For concreteness, let’s work in the case $p = 2$; the general case is not really different and will give you good practice working with Frobenius.

A first computation suggests that $1, x, x^{3/2}, x^{5/2}$ generate $R^{1/2}$ over $R$. Indeed, consider a monomial $x^{n/2} \in R^{1/2}$. Here, $n/2$ can be any non-negative half integer except $1/2$. If $n/2 = 0, 1, 3/2, 5/2$, then $x^{n/2}$ is clearly in the $R$-span of $\{1, x, x^{3/2}, x^{5/2}\}$. There are two further cases to consider:

(a) $n/2 \in \mathbb{N}$ and $n/2 \geq 2$: Then $x^{n/2}$ is in $R$, and so $x^{n/2} \cdot 1$ is in the $R$-span of $\{1, x, x^{3/2}, x^{5/2}\}$;

(b) $n/2 \geq 7/2$ but not in $\mathbb{N}$: then we can write $n = 2d + 3$ for some integer $d \geq 2$, so that $x^{n/2} = x^d \cdot x^{3/2}$ is in the $R$-span of $\{1, x, x^{3/2}, x^{5/2}\}$.

We conclude that $\{1, x, x^{3/2}, x^{5/2}\}$ is a generating set for $R^{1/p}$ over $R$.

Is it possible that $R^{1/p}$ is free over $R$ on these four generators? Well, in general, if some module over a ring is free, then localizing at any multiplicative system, the localized module is free of the same rank over the localization. Note that if $W$ is the multiplicative set of powers of $x^2$, then the inclusions of rings

$$W^{-1}R^{1/p} \subset (W^{-1}R)^{1/p} \subset (\mathbb{F}_p[x, x^{-1}])^{1/p} = \mathbb{F}_p[x^{1/p}, x^{-1/p}]$$

are all equalities. Indeed, since the powers of $x^{1/p}$ and $x^{-1/p}$ form an $\mathbb{F}_p$-basis for $\mathbb{F}_p[x^{1/p}, x^{-1/p}]$, we only have to check that both $x^{1/p}$ and $x^{-1/p}$ are in the ring $W^{-1}R^{1/p}$.

This is easy by direct computation:

$$x^{1/p} = \frac{1}{x^2} \cdot (x^{2p+1})^{1/p} \quad \text{and} \quad \frac{1}{x^{1/p}} = \frac{1}{x^2} \cdot (x^{2p-1})^{1/p} \in W^{-1}R^{1/p}. $$

Now since $\mathbb{F}_p[x^{1/p}, x^{-1/p}]$ is a localization of the free $R$-module $(\mathbb{F}_p[x])^{1/p}$, it is free of rank $p$ over $R$. So, if $R^{1/p}$ were to be free over $R$, it would have to be of rank two. That is, the elements $\{1, x, x^{3/2}, x^{5/2}\}$ can not be a free basis for $R^{1/p}$ over $R$.

Is it possible that there is some more clever set of two or three generators for the $R$-module $R^{1/p}$? We can use Nakayama’s lemma to see the answer is No. For this, we must reduce to the local case.

Observe that if $M$ is a module which can be generated by $d$ elements, then for any multiplicative set $W$, $W^{-1}M$ can also be generated by $d$ elements. Now let $W$ be the multiplicative set complementary to the maximal ideal $\langle x^2, x^3 \rangle$. The localization $W^{-1}R$ is a local ring (call it $S$) with maximal ideal generated by the images of $x^2$, $x^3$ (call it $m$), and residue field $S/m = \mathbb{F}_p$. By Nakayama’s Lemma (see [12] below), the minimal number of generators of the $S$-module $S^{1/p}$ is the $\mathbb{F}_p$-vector space dimension of

$$S/m \otimes_S S^{1/p} \cong S^{1/p}/mS^{1/p} = S^{1/p}/\langle x^2, x^3 \rangle S^{1/p}$$
which is spanned by the elements $1, x, x^{3/2}, x^{5/2}$, all of which are nonzero in the quotient. They are linearly independent over $\mathbb{F}_p$ because they are different degrees, so this vector space has dimension four. This means that $S^{1/p}$ needs four generators over $S$ and so $R^{1/p}$ needs at least (hence exactly) four generators as an $R$-module, too.

**Exercise 1.2.** If $R = \mathbb{F}_p[x^2, x^3]$, verify that $R^{1/p}$ is not a free $R$-module for any prime $p$. Find a minimal set of generators for $R^{1/p}$ over $R$.

The work we did in Example 1.11 made use of Nakayama’s Lemma, possibly the most important lemma in all of commutative algebra:

**Lemma 1.12.** (Nakayama’s Lemma) Let $R$ be a (not-necessarily Noetherian) local ring with maximal ideal $m$ and residue field $R/m = K$. Given a finitely generated $R$-module $M$, note that $M/mM$ is a finite dimensional vector space over $K$. Then a given set of elements $\{x_1, \ldots, x_t\} \subset M$ is a minimal generating set for $M$ if and only if their classes $\{\overline{x_1}, \ldots, \overline{x_t}\}$ in $M/mM$ are a $K$-vector space basis.

Intuitively, if we think of $M$ as a coherent sheaf on $\text{Spec} R$ and the elements $\{x_1, \ldots, x_t\} \subset M$ as sections, Nakayama’s Lemma says that these sections are a minimal generating set for $M$ in a neighborhood of a point $m$ if and only if, evaluating the sections at that point, we have a basis for the fiber $M/mM$ of $M$ over $m$. This is more or less obvious in the special case where $M$ is the sheaf of sections of a vector bundle (ie, locally free): any set of sections in a neighborhood of a point $p$ will generate the sheaf in a neighborhood of $p$ if and only if evaluating these sections at $p$ produces a basis for the fiber at $p$. So Nakayama’s Lemma says that the obvious condition for checking that sections of a vector bundle generate it at a point extends to the case of an arbitrary coherent sheaf.

Our computation in Example 1.11 generalizes to a formula for the minimal number of generators for $F_* R$ in general, for any local ring $(R, m, K)$, provided it is finite.

**Proposition 1.13.** Let $(R, m, K)$ be an arbitrary local ring of characteristic $p$. If $F_* R$ is a finitely generated module, then it is minimally generated by

$$[K : K^p] \dim_K (R/m^{[p]})$$

elements, where $m^{[p]}$ is the ideal of $R$ generated by the $p$-th powers of elements in $m$.

**Proof of Proposition 1.13.** By Nakayama’s Lemma, the minimal number of generators for $F_* R$ is the dimension of the $R$ module $R/m \otimes_R F_* R$. This is isomorphic to $F_* R/m F_* R \cong F_* (R/m^{[p]})$. Considered as a vector space over its residue field $F_*, K$, the ring $F_* (R/m^{[p]})$ has dimension equal to $\dim_K (R/m^{[p]})$. So to find its dimension over $K$ we must multiply by the degree of the field extension $K \to F_* K$. Remembering that $K \to F_* K$ is the $p$-th power map, we see that $[F, K : K] = [K : K^p]$, and the formula follows. □

**Notation 1.14.** For any ideal in any ring $R$ of characteristic $p$, the notation $I^{[p^e]}$ denotes the expansion of the ideal $I$ under the (iterated) Frobenius map $F^e : R \to R$. 

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sending \( r \mapsto r^{p^e} \). Explicitly, \( I^{[p^e]} \) is the ideal generated by the \( p^e \)-powers of all elements (or just any set of generators) of \( I \). Practicing the notation, note that

\[
I \cdot R^{1/p^e} = (I^{[p^e]} R)^{1/p^e} \quad \text{and} \quad I \cdot F_*^e R = F_*^e I^{[p^e]}.
\]

We formally record another fact we used in Example 1.11 stating, essentially, that Frobenius commutes with localization in every possibly sense:

**Lemma 1.15.** Suppose \( R \) is a ring of prime characteristic and \( W \) is a multiplicative set in \( R \). Then there is a natural \( W^{-1}R \)-module isomorphism

\[
W^{-1}F_*^e R \cong F_*^e(W^{-1}R)
\]

where the second \( F_*^e \) can be viewed as the Frobenius functor for either \( W^{-1}R \)-modules or for \( R \)-modules. Furthermore, this identification is also an isomorphism of rings.

Equivalently (when \( R \) is reduced), we have a natural identification \( W^{-1}R^{1/p^e} \cong (W^{-1}R)^{1/p^e} \) in the categories of rings or \( W^{-1}R \)-modules.

**Proof.** The basic point is that inverting all the elements of \( W \) is the same as inverting only their \( p \)-th-powers. More precisely, consider the natural map

\[
W^{-1}F_*^e R \rightarrow F_*^e(W^{-1}R) \quad \frac{1}{g} \cdot F_*^e r \mapsto F_*^e(r/g^{p^e}).
\]

It is certainly surjective since

\[
F_*(x/g) = F_*(xg^{p^e-1}/g^{p^e}) = \frac{1}{g} \cdot F_*^e xg^{p^e-1}.
\]

It is also easily verified to be linear in all relevant ways, and to respect the multiplication in the corresponding rings as well. Thus we simply need to check that it is injective. Hence suppose that \( F_*^e(r/g^{p^e}) = 0 \). This means that there exists \( h \in W \) such that \( hr = 0 \). But then

\[
\frac{1}{g} F_*^e(r) = \frac{h}{gh} F_*^e(r) = \frac{1}{gh} F_*^e(h^{p^e}r) = \frac{1}{gh} F_*^e(0) = 0
\]

in \( W^{-1}F_*^e R \) as well. \( \square \)

**Remark 1.16.** The same argument shows that for any \( R \)-module \( M \), we have a natural identification

\[
W^{-1}F_*^e M \rightarrow F_*^e(W^{-1}M)
\]

of \( W^{-1}R \)-modules. Of course, it follows that Frobenius behaves well with respect to localization for quasi-coherent sheaves on any scheme of characteristic \( p \).

**Definition 1.17.** A ring of prime characteristic is said to be \( F \)-finite if Frobenius is a finite map, i.e. if \( F_* R \) is a finitely generated \( R \)-module.

Most of the rings we encounter in algebraic geometry are \( F \)-finite:

**Exercise 1.3.** Let \( K \) be a field of characteristic \( p \), and assume \([K : K^p] = d < \infty\). Prove that

(a) For \( R = K[x_1, \ldots, x_n] \), the \( R \)-module \( F_* R \) is free of rank \( dp^n \).
(b) For $R = K[[x_1, \ldots, x_n]]$, the $R$-module $F_* R$ is free of rank $dp^n$.
(c) Any homomorphic image of an F-finite ring is F-finite. In particular, any finitely generated $K$ algebra is F-finite, and any complete local ring with residue field $K$ is F-finite.
(d) Any localization of an F-finite ring is F-finite.

2. Regular rings

Manifolds are boring locally: all points look more-or-less the same in their respective local neighborhoods. In algebraic geometry, however, there are lots of interesting differences between points on a given variety. Not everything is smooth, and we wind up studying a lot of “singularities,” a local issue. Commutative algebra can be viewed as “local” algebraic geometry. To talk about singularities, we need to review dimension and tangent spaces in commutative algebra.


Definition 2.1. The Krull dimension of a ring $R$ is defined to be the maximal length $n$ of a chain of prime ideals

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n \not\subset R.$$ 

More generally, given any prime ideal $Q$, the height $m$ of $Q$ is the maximal length of a chain of prime ideals

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_m = Q.$$ 

Example 2.2. Here are some examples of dimension.

(a) The dimension of a field $k$ is zero, as it has only one prime ideal $\langle 0 \rangle$. Notice that Spec $k$ is a single point.
(b) The dimension of a PID (such as $k[x]$) is one since nonzero prime ideals are incomparable. Notice that if $k = \overline{k}$ then Spec $k$ consists of closed points of the form $\langle x - \lambda \rangle$, where $\lambda$ ranges through all of $k$, and the zero ideal, which is dense in Spec $k[x]$. We call this dense point the generic point of Spec $k[x]$ since it is in every non-empty open set.
(c) The dimension of $k[x,y]$ is 2 assuming $k$ is a field. A maximal chain of prime ideals is $0 \subseteq \langle x \rangle \subseteq \langle x, y \rangle$.

We recall some basic facts about Krull dimension which can be found in any introductory source on commutative algebra, such as Hochster’s Notes. The heart of the matter is essentially Krull’s Principal Ideal Theorem, which states that the height of an $n$-generated ideal in a Noetherian ring is at most $n$. This is the key point needed to prove (c) below.

Proposition 2.3. Let $R$ be an arbitrary commutative ring.

(a) For any ideal $I$, $\dim R \geq \dim R/I$.
(b) For any multiplicative set $W$, $\dim R \geq \dim W^{-1}R$. 
If \((R, \mathfrak{m})\) is a Noetherian local ring and \(x \in \mathfrak{m}\), then \(\dim(R/\langle x \rangle) \geq \dim R - 1\), with equality if \(x\) is not in any minimal prime (e.g. not a zero divisor).  

If \((R, \mathfrak{m})\) is a Noetherian local ring, then \(\dim R\) is finite and
\[
\dim R = \text{least number of generators of an ideal } I \text{ with } \sqrt{I} = \mathfrak{m}.
\]

Observe that if \((R, \mathfrak{m})\) is any Noetherian local ring of dimension \(d > 0\), the Prime Avoidance Lemma allows us to chose \(x_1 \in \mathfrak{m}\) not in any minimal prime of \(R\). Thus by iterating statement (c), we can sequentially mod out a sequence of elements in \(\mathfrak{m}\), say \(x_1, \ldots, x_d \in \mathfrak{m}\), so that the dimension drops by exactly one at each step. In this case the elements \(x_1, \ldots, x_d\) are called a system of parameters for \(R\). That is, a system of parameters for a Noetherian local ring \((R, \mathfrak{m}, K)\) of Krull dimension \(d\) is any set of \(d\) elements of \(\mathfrak{m}\) such that, equivalently,

\[\begin{align*}
(a) & \text{ } \mathfrak{m} \text{ is a minimal prime of } \langle x_1, \ldots, x_d \rangle; \\
(b) & \text{ } \text{The radical of } \langle x_1, \ldots, x_d \rangle \text{ is } \mathfrak{m}; \\
(c) & \text{ } \text{The quotient } R/\langle x_1, \ldots, x_d \rangle \text{ has Krull dimension zero.} \\
(d) & \text{ } \text{For each } i, \text{ we have } \dim(R/\langle x_1, \ldots, x_i \rangle) = d - i.
\end{align*}\]

The prime avoidance argument above shows that systems of parameters always exist. If \(\mathfrak{m}\) is a minimal prime of \(R\), then \(\dim R = 0\) and the empty set is a system of parameters.

A harder theorem about dimension justifies why many authors use the word “codimension” instead of height:

**Theorem 2.4.** Let \(R\) be a domain of finite type over a field or a complete local Noetherian domain. For any \(Q \in \text{Spec } R\), we have
\[
\dim R = \text{height } Q + \dim R/Q.
\]

Theorem 2.4 holds much more generally than stated here, but perhaps surprisingly, not in general. Equality is closely related to the catenary condition on \(R\), which means that any two saturated chains of primes connecting two fixed primes of \(R\) has the same length. Nearly all rings that come up in algebraic geometry are catenary. This is discussed in detail, for example, in Matsumura’s book.

### 2.2. The Zariski Tangent Space.

**Setting 2.5.** From now on, the notation \((R, m, K)\) denotes a Noetherian local ring. Here, \(m\) is the unique maximal ideal and \(K = R/m\) is the residue field.

An important special case to keep in mind is the local ring of a point \(p\) on a variety over an algebraically closed field \(K\). Choosing an affine chart \(V\) of \(p\) and a closed embedding \(V \subseteq K^n\), we can make \((R, m, K)\) explicit as follows. Let \(K[V]\)

---

1. This sufficient condition on \(x\) is not necessary. For example, the two-dimensional ring \(K[[x,y]]_{(x^2, y^2)}\) has two minimal primes \((x, y)\) and \((z)\); the element \(x - y\) is in one of them, but killing it, the dimension drops by one, none-the-less.
be the coordinate ring of $V$ and $m_p$ the corresponding maximal ideal of all functions vanishing at $p$. Then $R$ is the localization $K[V]_{m_p}$. The “evaluation at $p$ map”

$$K[V] \to K, \quad \phi \mapsto \phi(p)$$

has kernel $m_p$ and hence can be identified with residue map $R \to R/m$.

**Definition 2.6.** The Zariski cotangent space of the Noetherian local ring $(R, m, K)$ is the $K$-vector space $m/m^2$. The Zariski tangent space is its dual $\text{Hom}_K(m/m^2, K)$. Its dimension is called the embedding dimension of $R$. By Nakayama’s Lemma, the embedding dimension can also be defined as the minimal number of generators for the maximal ideal $m$.

**Example 2.7.** Consider the point $p = (\lambda_1, \ldots, \lambda_n) \in K^N$. The corresponding maximal ideal $m_p$ of all polynomials vanishing at $p$ is $(x_1 - \lambda_1, \ldots, x_n - \lambda_n)$ and the local ring at $p$ is $R = K[x_1, \ldots, x_N]_{(x_1-\lambda_1,\ldots,x_n-\lambda_n)}$.

Each polynomial $\phi \in K[x_1, \ldots, x_N]$ can be expanded around $p$ as follows:

$$\phi = \phi(p) + \frac{\partial \phi}{\partial x_1}(p)(x_1 - \lambda_1) + \cdots + \frac{\partial \phi}{\partial x_N}(p)(x_N - \lambda_N) + \frac{1}{2!} \frac{\partial^2 \phi}{\partial x_1^2}(p)(x_1 - \lambda_1)^2 + \cdots$$

where the operators $(\frac{\partial^d}{\partial x_1^{a_1}} \circ \cdots \circ \frac{\partial^d}{\partial x_n^{a_n}})$ (with $d = a_1 + \cdots + a_n$) make sense over any field. Note that if $\phi$ vanishes at $p$, then $\phi$ has a first order term around $p$ that looks like

$$d_p \phi := \frac{\partial \phi}{\partial x_1}(p)(x_1 - \lambda_1) + \cdots + \frac{\partial \phi}{\partial x_N}(p)(x_N - \lambda_N).$$

This gives a $K$-linear map

$$m_p \to K^N, \quad \phi \mapsto d_p \phi \mapsto \left[ \frac{\partial \phi}{\partial x_1}(p), \ldots, \frac{\partial \phi}{\partial x_N}(p) \right].$$

The product rule for differentiation ensures that every element of $m_p^2$ is killed by this map, so there is a map of $K$-vector spaces

$$m_p/m_p^2 \to K^N, \quad \phi + m_p^2 \mapsto d_p \phi \mapsto \left[ \frac{\partial \phi}{\partial x_1}(p), \ldots, \frac{\partial \phi}{\partial x_N}(p) \right],$$

which is easily seen to be an isomorphism. This identifies each minimal generator $(x_i - \lambda_i) + m_p$ in $m_p/m_p^2$ with $d_p x_i$.

The Krull dimension of a local Noetherian ring $R$ is *at most* its embedding dimension (this follows immediately from the Krull Principal Ideal Theorem, or alternatively from Lemma 2.3(d)). When these dimensions are equal, intuitively, we have a tangent space which is a “good linear approximation” to our scheme $\text{Spec} R$ at $m$. This motivates the following definition.

**Definition 2.8.** A local Noetherian ring $(R, m, K)$ is **regular** if its Krull dimension is equal to its embedding dimension, or equivalently, if its maximal ideal can be generated by $\dim R$ elements.
(a) A regular local ring of dimension zero is the same as a field. Its maximal ideal is generated by no elements, hence must be the zero ideal.
(b) A regular local ring of dimension one is the same as a DVR. Its unique maximal ideal is generated by one element called a uniformizing parameter.
(c) The localization of the polynomial ring \( R = K[x_1, x_2, \ldots, x_n] \) at any maximal ideal of the form \( m_p = (x_1 - \lambda_1, \ldots, x_n - \lambda_n) \) is regular of dimension \( n \). The elements \( \frac{x_i - \lambda_i}{1} \) are minimal generators for the maximal ideal of \( R_{m_p} \).

**Proposition 2.10.** Every regular local ring is a domain.

**Proof.** Let \((R, m)\) be a regular local ring. First note that if \( \dim R = 0 \), then \( R \) is a field, hence certainly a domain. We proceed by induction on dimension.

Assume \( \dim R > 0 \). So \( m \) is not a minimal prime of \( R \), and by the Prime Avoidance Lemma, we can choose \( x \in m \setminus m^2 \) not in any minimal prime of \( R \). This means that the dimension of \( R/\langle x \rangle \) drops by one, as does the embedding dimension. The ring \( R/\langle x \rangle \) is thus a regular local ring of smaller dimension, so by induction, is a domain. This means that \( \langle x \rangle \) is a prime ideal of \( R \), so it must contain some minimal prime \( P \subset \langle x \rangle \). For any \( y \in P \), we can therefore write \( y = rx \) for some \( r \in R \), so that \( xr \in P \) and hence \( r \in P \). That is, \( P = \mathfrak{m}P \). Now the \( R \)-module \( P/\mathfrak{m}P \) and its quotient \( P/\mathfrak{m}P \) are zero, so that Nakayama’s Lemma implies also that \( P = 0 \). That is, \( R \) is a domain. \( \square \)

For future reference, we isolate a useful trick from the proof:

**Lemma 2.11.** Let \((R, m)\) be a Noetherian local domain and let \( x \in m \) be a non-zero element. If \( x \notin m^2 \), then \( R \) is regular if and only if \( R/\langle x \rangle \) is regular. Conversely, if both \( R/\langle x \rangle \) and \( R \) are regular, then \( x \notin m^2 \).

**Proof.** Our hypothesis ensures that killing \( x \), the ring \( R/\langle x \rangle \) has dimension equal to \( \dim R - 1 \). On the other hand, the embedding dimension drops by one if and only if \( x \in m \setminus m^2 \). \( \square \)

### 2.3. Regularity in the Classical Case.

Regularity for a local ring generalizes the property of smoothness for points on a variety over an algebraically closed field. Indeed, the local ring of a point on such a variety is regular if and only if the point is smooth. Before proving this, let us review tangent spaces and smoothness in the classical setting.

**Example 2.12.** Fix an algebraically closed field \( K \). Consider an irreducible hypersurface \( V = \mathcal{V}(f) \) in \( K^n \). From calculus, we know how to find the (affine) linear subspace of \( K^n \) best approximating \( V \) at a given point \( p \) on \( V \). We linearize \( f \) at \( p \), writing

\[
d_pf = \frac{\partial f}{\partial x_1}(p)d_px_1 + \cdots + \frac{\partial f}{\partial x_N}d_px_N = \frac{\partial f}{\partial x_1}(p)(x_1 - \lambda_1) + \cdots + \frac{\partial f}{\partial x_N}(x_N - \lambda_N)
\]

Note that the hypothesis “Noetherian” is embedded in the word “regular”.

More generally, \( R \) need not be a domain but then we should take \( x \) not in any minimal prime.
and look at the linear variety defined by \( d_p f \). This is the tangent space approximation to \( V \) at \( p \). Being defined by one linear equation, provided that equation is not zero, it is a linear space of dimension \( N - 1 \) and nicely approximates the \( N - 1 \) dimensional variety \( V \) near \( p \); in this case we say \( V \) is smooth at \( p \). However, if \( d_p f = 0 \), then the tangent space to \( V \) at \( p \) is all of \( K^N \), and is not a good approximation to \( V \) at \( p \); this is the non-smooth or singular case. In particular, the smooth locus of \( V(f) \) is open; its complement is the closed set defined by the vanishing of all partial derivatives of \( f \).

Let us look at the cone \( \mathbb{V}(x^2 + y^2 - z^2) \subset K^3 \) explicitly. Its tangent space at any point \( p = (a, b, c) \) on the cone is \( \mathbb{V}(d_p f) = \mathbb{V}(2a(x - a) + 2b(y - b) - 2c(z - c)) \subset K^3 \). This linear variety is two dimensional except when \( p = (0, 0, 0) \), where it is \( K^3 \).

The Zariski cotangent space of \( O_{V, p} \) at \( p = (a, b, c) \) is

\[
m_p/m_p^2 = (x - a, y - b, z - c)/(x - a, y - b, z - c)^2
\]

in the ring \( K[x, y, z]_{m_p}/((x^2 + y^2 - z^2) + m_p^2) \). At the origin, for example, this is the ring \( K[x, y, z]/(x, y, z)^2 \), and the space \( m_p/m_p^2 \) is the three dimensional space spanned by the classes of \( x, y, z \). But if \( p \) is not the origin, we have

\[
K[x, y, z]_{m_p} \cong K[(x - a), (y - b), (z - c)]/(2a(x - a) + 2b(y - b) - 2c(z - c)) + m_p^2.
\]

So, if \( c \neq 0 \), for example, then we can solve for \( (z - c) \) in terms of \( (x - a) \) and \( (y - b) \). This means that \( m_p/m_p^2 \) is spanned by \( (x - a), (y - b) \) (which we can also write \( d_px, d_py \)). More generally, the cotangent space to a smooth point of the cone will have basis consisting of two of the three elements \( d_px, d_py, d_pz \), depending on which coordinates of \( p \) are non-zero.

The previous example can be formalized into the so-called **Jacobian Criterion for Smoothness**. Let \( p \) be a point on an irreducible variety of dimension \( d \), and fix an affine chart (and closed immersion) so as to assume that \( p \in V \subset K^N \) is an affine algebraic variety defined by the radical ideal \( I_V \). We can pick generators for \( I_V \), say \( \{f_1, \ldots, f_l\} \subset K[x_1, \ldots, x_N] \).

The linear space most closely approximating \( V \) at \( p \) is

\[
T_p V = \mathbb{V}\{d_p f \}_{f \in I_V} = \mathbb{V}(d_pf_1, \ldots, d_pf_l).
\]

So \( T_p V \) at the point \( p = (\lambda_1, \ldots, \lambda_n) \) is the solution set of the system of linear equations

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_l}{\partial x_1} & \frac{\partial f_l}{\partial x_2} & \cdots & \frac{\partial f_l}{\partial x_n}
\end{bmatrix}
\begin{bmatrix}
x_1 - \lambda_1 \\
x_2 - \lambda_2 \\
\vdots \\
x_n - \lambda_n
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
\vdots \\
0 \end{bmatrix}.
\]

---

4Here, we assume \( \text{char } K \neq 2 \), for otherwise \( \mathbb{V}(x^2 + y^2 - z^2) \) is either the linear variety defined by \( x + y + z \) or the everywhere non-reduced scheme \( \mathbb{V}((x + y + z)^2) \), depending on our point of view. In the latter case, the argument here shows that \( d_p f = 0 \) for all \( p \) so that \( V \) is not smooth at any point.
A point \( p \) is a smooth point if and only if \( \dim T_p V = \dim_p V = d \). The Rank-Nullity Theorem from basic linear algebra implies that the dimension of \( T_p V \) is \( N - \text{rank } J_p \) where \( J_p \) denotes the Jacobian matrix \( \left[ \frac{\partial f_i}{\partial x_j} \right] \) at \( p \). Thus \( p \) is a smooth point if and only if \( N - \text{rank } J_p \) is equal to the dimension of \( V \) at \( p \). Since we know that the dimension of \( T_p V \) is at least \( \dim_p V \), we therefore have that \( p \) is a non-smooth point of the \( d \)-dimensional variety \( V \) if and only if \( N - \text{rank } J_p < N - d \), a condition described by the vanishing, at \( p \), of all \((N-d)\)-sized minors of the Jacobian matrix \( \left[ \frac{\partial f_i}{\partial x_j} \right] \).

Summarizing, the **Jacobian Criterion for Smoothness** says that: The locus of non-smooth points of the equidimensional variety \( V \subset K^N \) whose radical ideal is generated by \( f_1, \ldots, f_t \) is the closed subset

\[
V \cap \mathcal{V}(\text{all } c \times c \text{ sub-determinants of the Jacobian matrix } \left[ \frac{\partial f_i}{\partial x_j} \right]) \subset V
\]

where \( c = N - \dim V \).

**Example 2.13.** Consider \( k[x, y, z]/(x^2 y - z^2) \), a ring of dimension 2. The Jacobian matrix has only three entries, \( 2xy, x^2, 2z \). The singular locus of the variety \( \mathcal{V}(x^2 y - z^2) \) is defined by the ideal of \( 1 \times 1 \) minors of the Jacobian matrix, that is, is it the subset \( \mathcal{V}(2xy, x^2, 2z, x^2 y - z^2) \). Although the specific form of the equations for the singular locus is somewhat different in characteristic two, we none-the-less see that the singular set is \( \mathcal{V}(x, z) \) in all characteristics: it can be described as the “y-axis.”

With these calculations in mind, the equivalence of regularity and smoothness in the setting of classical algebraic geometry is easy:

**Proposition 2.14.** Let \( O_{V,p} \) be the local ring of a point \( p \) on an variety \( V \) over an algebraically closed field. Then \( O_{V,p} \) is regular if and only if \( p \) is a smooth point of \( V \).

**Proof of Proposition 2.14.** There is no loss of generality in choosing coordinates so that \( p \) is the origin in \( K^N \). In this case, the local ring of \( p \) on \( K^N \) is \( K[x_1, \ldots, x_N]_{(x_1, \ldots, x_N)} \) whose maximal ideal we denote by \( m_p \). The local ring of \( p \) on the closed subvariety \( V \) is \( O_{V,p} = K[x_1, \ldots, x_N]_{(x_1, \ldots, x_N)}/(f_1, \ldots, f_t) \), whose maximal ideal we denote \( \overline{m}_p \).

We have already seen that the map

\[
K[x_1, \ldots, x_N] \to K^N \quad \phi \mapsto d_p \phi = \left[ \frac{\partial \phi}{\partial x_1} \quad \frac{\partial \phi}{\partial x_2} \quad \cdots \quad \frac{\partial \phi}{\partial x_N} \right]^p
\]

is an isomorphism of local rings. The assumption that the field is algebraically closed here is crucial. See Example 2.17.
identifies $m_p/m_p^2$ with $K^N$ as $K$-vector spaces. Under this map, the vector space $I_V$ is sent to the span of $\{d_pf \mid f \in I_V\}$, which is the same as the span of $\{d_pf_1, \ldots, d_pf_t\}$. Thus there is an induced isomorphism:

$$\frac{m_p}{I_V + m_p^2} \to K^N/\text{span}\{d_pf_1, \ldots, d_pf_t\} \cong K - \text{dual of } \ker \begin{bmatrix} d_pf_1 \\ \vdots \\ d_pf_t \end{bmatrix} = \ker \left[ \frac{\partial f_i}{\partial x_j} \right]_p^*$$

But the left-side vector space is the Zariski cotangent space $m_p/m_p^2$. The condition for regularity of the local ring $O_{V,p}$ is thus the same as the condition of smoothness of $V$ at $p$, namely, that this dimension is $N - \dim O_{V,p} = N - \dim_p V$. □

**Remark 2.15.** The proof actually gives a *natural* isomorphism between the Zariski cotangent space of the local ring $O_{V,p}$ and the cotangent space of $V$ at $p$, that is, the space of linear functionals on the kernel of the Jacobian matrix. If you don’t want to fuss with the naturalness of this map, you can simply compare the dimensions of $m_p/(I_V + m_p^2)$ and $K^N/\text{span}\{d_pf_1, \ldots, d_pf_t\}$. The latter of this is clearly $N$ minus the rank of the Jacobian matrix. This completes the proof of the Proposition.

**Remark 2.16.** For simplicity, when we use the word “variety” in this course, we will always mean it in the classical context: the variety is assumed defined over an algebraically closed field $K$, and points are (classical, closed) points defined over that field $K$. We do not insist our varieties are irreducible, but since many arguments can be reduced to this case by considering components separately, we do so when convenient. Because $K$ is algebraically closed, the maximal ideal of a point $p$ will always be generated by $(x - \lambda_1, \ldots, x_n - \lambda_n)$ where $p = (\lambda_1, \ldots, \lambda_n)$ in some appropriate affine coordinates. If the field is not algebraically closed, of course, only the ideals of $K$-points have this form. Our proof of Proposition 2.14 is valid for $K$-points on any scheme of finite type over a field $K$.

**Example 2.17.** Consider $k = \mathbb{F}_p(t)$ and $R = k[x]/(x^p - t)$. Obviously $R$ is regular since it is a field $\cong k(t^{1/p})$. The scheme $\text{Spec } R$ is irreducible and of finite type over the field $k$, so it might be tempting to call it a “variety” over $k$. However, the field $k$ is not algebraically closed, so regularity and smoothness may not agree. Indeed, the Jacobian matrix has a single entry, $\frac{\partial}{\partial x}(x^p - t)$, which is zero. Hence the rank of the Jacobian matrix is 0, not 1, as we’d expect for a smooth variety. This regular scheme is of finite type over field $k$, but not smooth over $k$.

The notion of *smoothness* is a relative notion, meaning that it is actually a property of a map. The Jacobian criterion, properly interpreted, works quite generally as a criterion for *smoothness* of any morphism of schemes (after passing to affine charts). We will not pursue point of view this right now. Let us simply say that if

---

Here I am using the Math 217 fact that for any $t \times N$ matrix $J$ there is a natural isomorphism $K^N/\text{im}(J^*) \cong [\ker J]^*$. This follows by dualizing the exact sequences $0 \to \ker J \to K^N \xrightarrow{J^*} K^t$ which produces $0 \Leftarrow [\ker J]^* \Leftarrow [K^N]^* \xrightarrow{J^{**}} [K^t]^*$. Note that $\text{im}(J^*)$ is the row space of $J$; in our case, the rows of $J$ are the $d_pf_i$. 
2. REGULARITY AND KUNZ’S THEOREM

$K$ is an algebraically closed field and $V$ is a variety over $K$, then the structure map $V \to \text{Spec } K$ is smooth at a point $p \in V$ if and only if the variety $V$ is smooth at $p$, which as we have proved, is equivalent to saying the local ring $O_{V,p}$ is a regular local ring.

2.3.1. Derivations and the Tangent Space. If you have studied some differential geometry, you know that the tangent space at a point of a manifold can be described as the space of derivations at a point. This is true for algebraic varieties too: the Zariski tangent space is naturally identified with the space of derivations at a point.

Definition 2.18. Let $p$ be a $K$-point on an algebraic variety $V$ over $K$. The $K$-derivations at $p$ is the $K$-vector space

$$\text{Der}_K(\mathcal{O}_{V,p}, K) = \{ \delta : \mathcal{O}_{V,p} \to K \mid \delta \text{ is } K\text{-linear, and } \forall f, g, \in \mathcal{O}_V, \delta(fg) = f(p)\delta(g) + g(p)\delta(f) \}.$$  

The condition $\delta(fg) = f(p)\delta(g) + g(p)\delta(f)$ for all $f$ and $g$ is called a Leibnitz condition. It implies that every derivation $\delta$ kills $m_p^2$, since if $f, g \in m_p^2$, then

$$\delta(fg) = f(p)\delta(g) + g(p)\delta(f) = 0.$$  

Thus in the definition, we can replace the local ring $\mathcal{O}_{V,p}$ by $\mathcal{O}_{V,p}/m_p^2$ or by the coordinate ring of any affine neighborhood of $p$.

Proposition 2.19. Let $(R, m, K)$ be the local ring of a $K$-point $p$ on a variety $V$. Then the space of $K$-derivations at $p$ is canonically identified with the Zariski tangent space at $p$.

Proof. Given $\delta \in \text{Der}_K(R, K)$, we have already observed that $\delta|_{m^2} = 0$. Thus, restricting to $m$ gives a $K$-linear map

$$\text{Der}_K(R, K) \to \text{Hom}_K(m/m^2, K) \quad \delta \mapsto \overline{\delta|m} : m/m^2 \to K.$$  

We prove that this $K$-linear map is an isomorphism by explicitly constructing an inverse: Given $\overline{\delta} \in \text{Hom}_K(m/m^2, K)$, we can lift to $\delta : m \to K$. Now define a $K$-linear map $\Delta$ by the composition

$$R \to m \xrightarrow{\delta} K \quad f \mapsto f - f(p) \mapsto \delta(f - f(p)).$$  

It is straightforward to check that $\Delta$ is a $K$-derivation: we must check that

$$\delta(fg - f(p)g(p)) = g(p)\delta(f - f(p)) + f(p)\delta(g - g(p)).$$  

This follows from the fact that $(f - f(p))(g - g(p)) \in m_p^2$, hence is killed by $\delta$. □
3. Homological Characterizations of Regularity

In this section, we will prove the following foundational theorem of Serre and Auslander-Buchsbaum:

**Theorem 3.1.** Let \((R, m, K)\) be a local Noetherian ring. Then the following conditions are equivalent.

(a) \(R\) is regular.

(b) The residue field \(K = R/m\) has a finite free resolution.

(c) Every \(R\)-module has a finite free resolution.

Moreover, the projective dimension of \(K\) is equal to the Krull dimension of \(R\), and bounds the projective dimension for every other \(R\)-module.

Theorem 3.1 was an important breakthrough in the 1950's. The idea to generalize classical algebraic geometry to more general settings we now know as schemes had been developing for some time. Zariski had introduced his tangent space \((m/m^2)^*\) as a generalizable description of the tangent space already in 1947, and regular rings as an analog of smooth varieties were emerging. However, clearly any reasonable generalization of smoothness for varieties ought to be an open condition, and it was not clear that regular rings had this property. Indeed, it was not even known at the time that the localization of a regular local ring at a prime ideal was a regular local ring!

Fortunately, Theorem 3.1 clarified the situation with the following important consequence:

**Corollary 3.2.** If \(R\) is a regular local ring, and \(Q\) is a prime ideal of \(R\), then \(R_Q\) is regular.

**Proof.** Since \(R\) is regular, \(R/Q\) has a finite \(R\)-free resolution by \(R\)-modules. Tensoring with the flat \(R\)-module \(R_Q\), we obtain a finite \(R_Q\)-free resolution of the residue field \((R/Q)_Q \cong R_Q/QR_Q\) of \(R_Q\). □

Corollary 3.2 ensures that a Noetherian ring has the property that its localization at every prime ideal is regular if and only if its localization at every maximal ideal is regular. A Noetherian ring with these equivalent properties is called regular. Likewise, a scheme is regular if and only if each of its stalks (or equivalently, stalks at closed points) is regular.

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7 Strictly speaking, we will prove only that every finitely generated module has a finite free resolution in (c); the general case follows fairly easily from that one. Or see Lemma 2 in §19 of Matsumura.

8 This turns out to be false! See Hoc. However, under mild hypothesis the regular locus is open, including F-finiteness. See Proposition 4.3 in Chapter 3.
3.1. Projective, Flat and Free Modules. Before proving Theorem 3.1, we detour through a review of flat and projective modules. The punchline is that for finitely generated modules over a Noetherian local ring, the conditions flat, projective and free are all equivalent.

**Definition 3.3.** Let $R$ be an arbitrary ring. An $R$-module $P$ is **projective** if the functor $\text{Hom}_R(P, -)$ is (right) exact. An $R$-module $P$ is **flat** if the functor $P \otimes_R -$ is (left) exact.

It is easy to see that free modules are both flat and projective. Under mild hypothesis, we have partial converses:

**Proposition 3.4.** If $M$ is a projective module over a local ring $(R, m)$, then $M$ is free.

**Proof.** We only prove the case when $M$ is finitely generated, the general case is hard and due to Kaplansky [Kap58].

Let $n = \dim_{R/m} M/mM$. By Nakayama’s lemma, we have a surjection $\kappa : R^\oplus n \twoheadrightarrow M$. Since $M$ is projective, we have a map $\sigma : M \rightarrow R^\oplus n$ so that $\kappa \circ \sigma : M \rightarrow R^\oplus n \rightarrow M$ is the identity (and in particular $\sigma$ is injective). It follows that

$$M/mM \xrightarrow{\sigma} (R/m)^{\oplus n} \xrightarrow{\kappa} M/mM$$

is also the identity. Thus $\sigma$ must also be an isomorphism (since it is an injective map between vector spaces of the same dimension). Hence by Nakayama’s lemma, $\sigma : M \rightarrow R^\oplus n$ is also surjective. But thus $\sigma$ is both injective and surjective and hence an isomorphism, which proves that $M$ is free. \(\square\)

**Proposition 3.5.** Let $M$ be a finitely presented module over an arbitrary ring $R$. If $M$ is flat, then $M$ is projective.

The finiteness assumption on $M$ is essential in Proposition 3.5: the localization $R_P$ is an example of a flat $R$ module, but it is rarely projective. However, the ring $R$ need not be Noetherian.

Before proving Proposition 3.5, we need to prove that flatness can be checked locally:

**Lemma 3.6.** Let $M$ be an arbitrary module over an arbitrary ring $R$. Then $M$ is flat if and only if $M_m$ is a flat $R_m$-module for every maximal ideal $m \subseteq R$.

**Proof.** It follows immediately from the definitions that if $M$ is a flat $R$-module, then $M_P$ is an flat $R_P$-module for all $P \in \text{Spec } R$.

For the converse, suppose that $0 \rightarrow A \rightarrow B$ is an injection of $R$-modules. Consider $K = \ker(A \otimes M \rightarrow B \otimes M)$, we will show that $K = 0$. Consider the exact sequence

$$0 \rightarrow K \rightarrow A \otimes M \rightarrow B \otimes M,$$

since localization is exact, for every maximal ideal $m \subseteq R$, we have that

$$0 \rightarrow K \otimes R_m \rightarrow A \otimes M \otimes R_m \rightarrow B \otimes M \otimes R_m,$$
is exact. But this is the same as saying that
\[ 0 \to K_m \to A_m \otimes M_m \to B_m \otimes M_m \]
is exact. Since \( M_m \) is flat by hypothesis, this implies that \( K_m = 0 \), and this holds for every maximal ideal. Thus \( K = 0 \). \( \square \)

**Proof of Proposition 3.5.** Since localization commutes with \( \text{Hom} \) from finitely presented modules, and exactness may be checked locally, we may assume that \((R, \mathfrak{m})\) is a local ring. In this case, we will show that \( M \) is free. Choose a minimal generating set for \( M \) and the corresponding surjection \( R^{\oplus t} \to M \to 0 \) with kernel \( K \) (which is finitely generated). We tensor with \( R/\mathfrak{m} \) to obtain
\[ \text{Tor}_1(M, R/\mathfrak{m}) \to K \otimes R/\mathfrak{m} \to (R/\mathfrak{m})^{\oplus t} \xrightarrow{\alpha} M \otimes R/\mathfrak{m} \to 0 \]
Since we picked a minimal generating set for \( M \), \( \alpha \) is bijective. Since \( M \) is flat, \( \text{Tor}_1(M, R/\mathfrak{m}) = 0 \) and hence \( K \otimes R/\mathfrak{m} = \mathfrak{m}K = 0 \) and so \( K = \mathfrak{m}K \). By Nakayama’s lemma this implies that \( K = 0 \). \( \square \)

Summarizing the above propositions in a useful compact form, we have the following key fact.

**Corollary 3.7.** Let \( M \) be a finitely generated module over a Noetherian local ring. Then the following are equivalent:

(a) \( M \) is projective;
(b) \( M \) is flat;
(c) \( M \) is free.

**3.2. Projective Resolutions.** Over any ring \( R \), an arbitrary module has a projective resolution. That is, given \( M \), there is a (usually infinite) exact sequence
\[ \cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0, \]
such that all of the \( P_i \) are projective.

In fact, we may easily construct a resolution in which every \( P_i \) is free. Choose a set of generators \( \{u_\lambda\}_{\lambda \in \Lambda} \) for \( M \), and let \( P_0 = \bigoplus_{\lambda \in \Lambda} Rb_\lambda \) on a correspondingly indexed set of generators \( \{b_\lambda\}_{\lambda \in \Lambda} \). There is a unique \( R \)-linear map \( P_0 \to M \) sending \( b_\lambda \mapsto u_\lambda \) for all \( \lambda \in \Lambda \). The kernel \( M_1 \) of \( P_0 \to M \) is referred to as a **first module of syzygies** of \( M \), and there is an exact sequence \( 0 \to M_1 \to P_0 \to M \to 0 \). Repeating this process for \( M_1 \), we find a free module \( P_1 \) mapping surjectively to \( M_1 \), which gives us an exact sequence \( 0 \to M_2 \to P_1 \to M_1 \to 0 \). These two exact sequences can be spliced together to form an exact sequence \( 0 \to M_2 \to P_1 \to P_0 \to M \to 0 \) where the map \( P_1 \to P_0 \) is the composition of the maps \( P_1 \to M_1 \) and \( M_1 \to P_0 \). The module \( M_2 \) is said to be a **second module of syzygies** for \( M \).
Recursively, we may form short exact sequences 0 → $M_n$ → $P_{n-1}$ → $M_{n-1}$ → 0 for all $n \geq 1$ (where $M_0 = M$), and splice together to get an exact sequence

\[(*)\quad 0 \rightarrow M_n \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow \cdots \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0\]

where the $P_i$ are free modules; the module $M_n$ is called an $n$th module of syzygies of $M$. The (usually infinite) sequence

\[(**)\quad \cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0\]

is a projective resolution of $M$.

**Definition 3.8.** The **projective dimension** of a module $M$ over a ring $R$, denoted $\text{Projdim}(M)$, is the length of the shortest possible projective resolution of $M$.

Note that the projective dimension can be infinite, and that it is zero if and only if $M$ is a non-zero projective module. By convention, the projective dimension of the zero module is $-1$.

If $R$ is Noetherian and $M$ is finitely generated, the free module $P_0$ mapping onto $M$ in our construction above may be taken to be finitely generated. In this case, the kernel $M_1$ is also finitely generated, so the module $P_1$ can again be assumed finitely generated. Inductively, the free modules $P_i$ may all be chosen to be finitely generated, so that $M$ has a free resolution by finitely generated free modules.

### 3.2.1. Minimal free resolutions over local rings

Let $(R, m, K)$ be a Noetherian local ring. In this case, a projective resolution is the same as a free resolution. In constructing a free resolution for a finitely generated $R$-module $M$ as above, we may insist that at every stage, we choose a minimal set of generators for $M_{n-1}$ and so construct a free $R$-module $P_{n-1}$ of smallest possible rank mapping onto $M_{n-1}$. A resolution constructed in this way is called a **minimal** free resolution of $M$. Equivalently, a projective resolution

\[\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0\]

is **minimal** if and only if one of the following equivalent conditions holds:

(a) Every $P_i$ has a free basis that maps to a minimal set of generators of $M_n$;

(b) $M_i \subset mP_{i-1}$ for all $i$. Indeed otherwise, there is generator of $M_i$ (the module of relations on the generators for $M_{i-1}$) that has a unit for the coefficient of some generator of $e_j$ of $P_{i-1}$; this gives us a relation on the generators of $M_{i-1}$ that can be be used to eliminate that generator from the presentation of $M_{i-1}$—that is, the chosen generating set was not minimal.

(c) For each map $d_i : P_i \rightarrow P_{i-1}$ between free modules in the resolution, the entries of the matrix representing $d_i$ have all entries in $m$.

It is not *a priori* obvious that a minimal free resolution is necessarily a shortest possible resolution. However, this follows from the next proposition, which gives a formula for projective dimension of finitely generated modules using Tor.
3. Using Tor to Detect Projective Dimension.

**Proposition 3.9.** Let $M$ be a finitely generated module over a Noetherian local ring $(R, m, K)$, and let $P_\bullet$ be a minimal free resolution. The modules $\text{Tor}_i(M, K)$, for $i \geq 0$, are finite-dimensional vector spaces over $K$ of dimension equal to the rank of the $i$-th free module in $P_\bullet$. Moreover the following conditions on $M$ are equivalent:

(a) The module $P_{n+1}$ in $P_\bullet$ is zero.
(b) The projective dimension of $M$ is at most $n$.
(c) $\text{Tor}_{n+1}(M, K) = 0$.
(d) $\text{Tor}_i(M, K) = 0$ for all $i \geq n + 1$.

In particular, the projective dimension of $M$ is $\sup \{n | \text{Tor}_n^R(M, K) \neq 0\}$, and the minimal free resolution of $M$ is a shortest possible projective resolution of $M$. In particular, $M$ has finite projective dimension if and only if its minimal free resolution is finite.

**Remark 3.10.** Proposition 3.9 is false for infinitely generated modules. For example, any module $M$ which is flat but not projective, such as the fraction field of a domain, say, will satisfy $\text{Tor}_i(M, K) = 0$ for all $i \geq 1$.

**Proof of Proposition 3.9.** Take a minimal free resolution $P_\bullet$ of $M$. Because all the entries of every matrix representing the maps $d_j$ in this resolution are in $m$, we see that after tensoring with $R/m$, all the maps in $P_\bullet \otimes_R K$ become 0 and each $P_i \otimes K$ is a vector space over $K$ whose dimension is the same as the rank of $P_i$. Hence, the homology of $P_\bullet \otimes_R K$ at the $i$th spot is $P_i \otimes K$, and the first statement follows. It is clear that (a) implies (b) implies (c), and then the result about Tor tells us that (c) implies (a). It is also clear that (a) implies (d) implies (c), and again, (c) implies (a). Thus the four conditions are all equivalent. There cannot be a projective resolution shorter than the minimal resolution, since $\text{Tor}_j(M, K)$ can be computed from any resolution, and we know that $\text{Tor}_n(M, K)$ is not zero when the minimal resolution has length $n$ (or more). \[\square\]

**Corollary 3.11.** For any Noetherian local ring $(R, m, K)$, the projective dimension of any finitely generated $R$-module is bounded above by the projective dimension of $K$.

**Proof.** This uses the non-trivial fact that $\text{Tor}_n^R(A, B) = \text{Tor}_n^R(B, A)$ in general, for arbitrary modules $A$ and $B$ over an arbitrary commutative ring $R$. So we can compute $\text{Tor}_i(M, K) = 0$ from a projective resolution of $K$ to see that all Tor’s vanish beyond proj dim $K$. Now invoke Proposition 3.9. \[\square\]

**Remark 3.12.** Let $(R, m, K)$ be a regular local ring. If you know about the Koszul complex, you might know that the Koszul complex on a minimal set of generators for $m$ gives a projective resolution of the residue field $K$ over $R$. This is one way to see that modules over a regular local ring of dimension $d$ have projective dimension at most $d$. We present a different proof below.
### 3.4. The Easy Direction of Serre’s Theorem

We can now give a straightforward inductive proof of the following theorem:

**Theorem 3.13.** If \((R, m, K)\) is a regular local ring, then the projective dimension of each finitely generated \(R\)-module is bounded above by \(\dim R\).

**Lemma 3.14.** Let \(R\) be a ring and let \(x \in R\) an element.

(a) Let \(Q_\bullet\) be an exact sequence of modules

\[
\cdots \to Q_{n+1} \to Q_n \to Q_{n-1} \to \cdots
\]

(possibly doubly infinite) with the property that \(x\) is a non-zerodivisor on all of the modules \(Q_n\). Then the complex \(Q_\bullet\) obtained by applying \(\_ \otimes R/xR\), which we may alternatively describe as

\[
\cdots \to Q_{n+1}/xQ_{n+1} \to Q_n/xQ_n \to Q_{n-1}/xQ_{n-1} \to \cdots,
\]

is also exact.

(b) If \(x\) is a non-zerodivisor in \(R\) and is also a nonzerodivisor on the module \(M\), while \(xN = 0\), then for all \(i\), \(\text{Tor}_i^R(M, N) \cong \text{Tor}_i^{R/xR}(M/xM, N)\).

**Proof.** (a) Because \(x\) is a non-zero divisor on every \(Q_n\), there is a short exact sequence of complexes

\[
0 \to Q_\bullet \xrightarrow{x} Q_\bullet \to \overline{Q}_\bullet \to 0
\]

which, at the \(n\)th spots, is \(0 \to Q_n \xrightarrow{x} Q_n \to Q_n/xQ_n \to 0\). The snake lemma yields an exact sequence

\[
\cdots \to H_n(Q_\bullet) \to H_n(Q_\bullet) \to H_n(\overline{Q}_\bullet) \to H_{n-1}(Q_\bullet) \to \cdots,
\]

and since \(H_n(Q_\bullet)\) and \(H_{n-1}(Q_\bullet)\) both vanish, so does \(H_n(\overline{Q}_\bullet)\).

(b) Consider a free resolution

\[
\cdots \to P_n \to \cdots \to P_0 \to M \to 0
\]

for \(M\). By part (a), this remains exact when we apply \(\_ \otimes R/xR\), which yields a free resolution of \(M/xM\) over \(R/xR\). Since \(x\) kills \(N\), \((P_\bullet \otimes R/xR) \otimes R/xR \cong P_\bullet \otimes R N\). The conclusion follows.

**Proof of Theorem 3.13** Let \((R, m, K)\) be a regular local ring. If \(\dim(R) = 0\), then the maximal ideal of \(R\) is generated by 0 elements. So \(R\) is a field, and every \(R\)-module is free (projective dimension zero).

Now suppose \(\dim(R) = d \geq 1\). Because regular rings are domains, any chosen \(x \in m \setminus m^2\) is a non-zero-divisor on \(R\). Given any finitely generated \(R\)-module \(M\), we can form an exact sequence

\[
0 \to M_1 \to P \to M \to 0
\]

where \(P\) is free and \(M_1\) is finitely generated. It suffices to show that \(M_1\) has a finite projective resolution of length at most \(d - 1\), since the resolution of \(M_1\) can splice
3. HOMOLOGICAL CHARACTERIZATIONS OF REGULARITY

Because \( M_1 \) is a submodule of the free module \( P \), note that \( x \) is a non-zero-divisor on \( M_1 \). Now by Lemma 3.14, we have \( \text{Tor}_n^R(M_1, K) \cong \text{Tor}^{R/xR}_n(M_1/xM_1, K) \) for all \( n \). By induction, the \((d-1)\)-dimensional regular local ring \( R/\langle x \rangle \) has the property that all finitely generated modules have projective dimension at most \( d-1 \), which means that \( \text{Tor}^{R/xR}_{d+1}(M_1, K) \cong \text{Tor}^R_d(M_1/xM_1, K) = 0 \). Proposition 3.9 then implies that the \( R \)-module \( M_1 \) has projective dimension at most \( d-1 \), so \( M \) has projective dimension at most \( d \).

3.5. The Difficult Direction. The converse is more difficult. We first prove a lemma about the behavior of projective dimension in exact sequences.

**Lemma 3.15.** Let \((R, m, K)\) be a Noetherian local ring, and consider a short exact sequence

\[
0 \rightarrow M_2 \rightarrow M_1 \rightarrow M_0 \rightarrow 0
\]

of finitely generated \( R \)-modules. If any two have finite projective dimension over \( R \), so does the third. Moreover:

(a) \( \text{Projdim} M_1 \leq \max \{ \text{Projdim} M_0, \text{Projdim} M_2 \} \).

(b) \( \text{Projdim} M_0 \leq \max \{ \text{Projdim} M_1, \text{Projdim} M_2 + 1 \} \).

(c) If \( M_0 \) and \( M_1 \) have finite projective dimension, then

i. \( \text{Projdim} M_2 = \text{Projdim} M_0 - 1 \) if \( \text{Projdim} M_1 < \text{Projdim} M_0 \); and

ii. \( \text{Projdim} M_2 \leq \text{Projdim} M_1 \) otherwise.

**Proof.** Consider the long exact sequence for Tor:

\[
\cdots \rightarrow \text{Tor}^R_{n+1}(M_1, K) \rightarrow \text{Tor}^R_{n+1}(M_0, K) \rightarrow \text{Tor}^R_{n}(M_2, K) \rightarrow \text{Tor}^R_{n}(M_1, K) \rightarrow \text{Tor}^R_{n}(M_0, K) \rightarrow \cdots
\]

If two of the \( M_i \) have finite projective dimension, then two of any three consecutive terms are eventually 0, and this forces the third term to be 0 as well.

The statements in (a), (b), and (c) bounding some \( \text{Projdim} M_j \) above for a certain \( j \in \{0, 1, 2\} \) all follow by looking at trios of consecutive terms of the long exact sequence such that the middle term is \( \text{Tor}^R_n(M_j, K) \). For \( n \) larger than the specified upper bound for \( \text{Projdim} R M_j \), the Tor on either side vanishes. The equality in (c) follows because with \( n = \text{Projdim} M_0 - 1 \), \( \text{Tor}^R_{n+1}(M_0, K) \) injects into \( \text{Tor}^R_{n}(M_2, K) \). \( \square \)

We also need the following linear algebra lemma.

**Lemma 3.16.** Let \((R, m, K)\) be a local ring, and let \( A, B \) be \( n \times n \) matrices over \( R \) and suppose that \( AB = xI_n \) for some \( x \in m - m^2 \). If every entry of \( A \) is in \( m \), then \( B \) is invertible.
Proof. We use induction on $n$. If $n = 1$, we have that $(x) = (a)(b) = (ab)$, where $a \in m$. Since $x \notin m^2$, we must have that $b$ is a unit.

Now suppose that $n > 1$. If every entry of $B$ is in $m$, the fact that $xI_n = AB$ implies that $x \in m^2$ again. Thus, some entry of $B$ is a unit. We can do invertible row and column operation on $B$ to bring it into block form

$$B' = \begin{pmatrix} 1 & \mathbf{0}_{1 \times (n-1)} \\ \mathbf{0}_{(n-1) \times 1} & B_0 \end{pmatrix},$$

where the notation $\mathbf{0}_{a \times b}$ indicates a zero matrix of size $a \times b$. Each of the row (respectively, column) operations has the effect of multiplying on the left (respectively, right) by an invertible $n \times n$ matrix. Thus, there are invertible $n \times n$ matrices $U$ and $V$ over $R$ such that

$$B' = UBV.$$

Now, with $A' = V^{-1}AU^{-1}$, we have

$$A'B' = V^{-1}AU^{-1}UBV = V^{-1}(AB)V = V^{-1}(xI_n)V = x(V^{-1}I_nV) = xI_n,$$

and note that $A'$ still has all entries in $m$. We can write

$$A' = \begin{pmatrix} a & \rho \\ \gamma & A_0 \end{pmatrix}$$

where $a \in R$ is a $1 \times 1$ matrix over $R$, and $\rho$ is $1 \times (n - 1)$, $\gamma$ is $(n - 1) \times 1$, and $A_0$ is $(n - 1) \times (n - 1)$, respectively. Then

$$xI_n = A'B' = \begin{pmatrix} a & \rho \\ \gamma & A_0 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}_{1 \times (n-1)} \\ \mathbf{0}_{(n-1) \times 1} & B_0 \end{pmatrix} = \begin{pmatrix} a & \rho B_0 \\ \gamma & A_0 B_0 \end{pmatrix}$$

from which we can conclude that $xI_{n-1} = A_0 B_0$. By the induction hypothesis, $B_0$ is invertible, and so $B'$ is invertible, and the invertibility of $B$ follows as well. This completes the proof of Theorem 3.18 and hence the proof of Serre’s theorem. □

Remark 3.17. An alternate proof was suggested during class, which does not appear to work. The idea was that if $AB = xI_n$, then $\det A \det B = x^n$, and since $\det A \in m^n$, we would know $\det B \notin m$ if we have that $x^n \notin m^{n+1}$. Unfortunately, however, it is not true that $x \in m \setminus m^2$ implies $x^n \in m^n \setminus m^{n+1}$. For example, in the ring $R = k[x,y,z]/(x^2 - y^{17}z^{23})$, the element $x \in m \setminus m^2$. But $x^2 \in m^{40} \subset m^3$.

Using Lemmas 3.15 and 3.16 we can prove the following technical theorem critical in proving that if $R$ has finite projective dimension over $(R,m,K)$ then $R$ is regular.

Theorem 3.18. Suppose that $(R,m,K)$ is a local Noetherian ring and that $M$ is a finitely generated $R$-module of finite projective dimension. If $x \in m - m^2$ is a non-zero divisor on $R$ but $xM = 0$, then $M$ has finite projective dimension over $R/xR$.

Before proving Theorem 3.18 we complete the proof of the harder direction of Serre’s theorem characterizing regular local rings as those with finite global dimension:
Theorem 3.19. Let \((R, m, K)\) be a Noetherian local ring such that \(\text{Projdim}_R K\) is finite. Then \(R\) is regular.

Proof. First consider the case where every element of \(m\) is a zero-divisor on \(R\), that is, where \(m\) is an associated prime of \(R\). Then there is some \(x \in R\) that is annihilated by \(m\). Now consider a finite minimal free resolution of \(K\):
\[
0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to R \to K \to 0.
\]
For \(i \geq 1\), the maps \(P_i \to P_{i-1}\) are given by matrices all of whose entries are in \(m\). Thus none of them can be injective: indeed, the element \(xe_1\) (where \(e_1\) denotes a standard unit column vector in any of the free modules \(P_i\)) will be sent to zero. In particular, the last map \(P_n \to P_{n-1}\) cannot be injective, a contradiction. We conclude that this finite free resolution must have had length 0, so that \(K \cong R\) is field, and hence regular.

Now suppose that \(m\) is not an associated prime of \(R\). Then we can choose \(x \in m \setminus m^2\) but not in any associated prime of \(R\) by the prime avoidance lemma. That is, we can choose \(x \in m \setminus m^2\) a non-zero divisor. By Theorem 3.18, the fact that \(K\) has finite projective dimension over \(R\) implies that it has finite projective dimension over \(R/xR\). By the induction hypothesis, \(R/xR\) is regular. Since \(x \notin m^2\) and \(x\) is not a zerodivisor, we conclude that \(R\) is regular (Lemma 2.11). This completes the proof of Serre’s Theorem, once we have established Theorem 3.18. □

It remains to prove Theorem 3.18.

Proof of Theorem 3.18. We may assume \(M\) is not 0. Likewise, \(M\) cannot be free over \(R\), since \(xM = 0\). Thus, we may assume \(\text{Projdim}_R M \geq 1\). We want to reduce to the case where \(\text{Projdim}_R M = 1\).

If \(\text{Projdim}_R M > 1\), we can think of \(M\) as a module over \(R/xR\) and define a map \((R/xR)^{\oplus h} \to M\) for some \(h\). It suffices to prove that the kernel, call it \(M_1\), has finite projective dimension over \(R/xR\). Because \(x\) is a non-zero-divisor, we have a short exact sequence
\[
0 \to R \xrightarrow{x} R \to R/xR \to 0,
\]
showing that \(R/xR\), and hence \((R/xR)^{\oplus h}\), have projective dimension one over \(R\). By Proposition 3.15 (c), \(\text{Projdim}_R M_1 = \text{Projdim}_R M - 1\). So since \(M_1\) is also a finitely generated \(R\) module killed by \(x\), by repeating if necessary with higher syzygy modules, we have reduced to the case where \(\text{Projdim}_R M = 1\).

Consider a minimal free resolution of \(M\) over \(R\), which will have the form
\[
0 \to R^n \xrightarrow{A} R^k \to M \to 0
\]
where \(A\) is an \(k \times n\) matrix with entries in \(m\). If we localize at \(x\), we have \(M_x = 0\), and so
\[
0 \to R^n_x \xrightarrow{A} R^k_x \to 0
\]
is exact. Thus, \(k = n\), and \(A\) is \(n \times n\). Let \(e_j\) denote the \(j\)th column of the identity matrix \(I_n\). Since \(M \cong R^n/\text{im} A\) is killed by \(x\), we see that every \(xe_j\) is in the image
of $A$, and so we can write $xe_j = Ab_j$ for a some $n \times 1$ column matrix $b_j$ over $R$. Let $B$ denote the $n \times n$ matrix over $R$ whose columns are $b_1, \ldots, b_n$. Then $xI_n = AB$. By the preceding Lemma, $B$ is invertible, and so $A$ and $AB = xI_n$ have the same cokernel, up to isomorphism. But the cokernel of $xI_n$ is $(R/xR)^{\oplus n}$, which is free over $R/xR$. So $M$ is a free $R/xR$ module, and hence certainly has finite projective dimension over $R/xR$. □

3.6. Completion and Regularity. We briefly review completion in the special case it is most used; all of the following facts can be found in Atiyah-MacDonald, Matsumura §29, or Mel’s notes.

Let $(R, m, K)$ be a Noetherian local ring. By definition, the completion $\hat{R}$ is defined as the inverse limit

$$\hat{R} := \varprojlim R/m^n$$

The completion $\hat{R}$ is also a local Noetherian ring of the same Krull dimension as $R$, with maximal ideal $\hat{m}$ and residue field canonically isomorphic to $K$.

Example 3.20. Consider the polynomial ring $k[x_1, \ldots, x_n]$ localized at the maximal ideal $m = \langle x_1, \ldots, x_n \rangle$ defining the origin in $k^n$. Then the completion is $\hat{R} = k[x_1, \ldots, x_n]$, the formal power series in the $x_i$. Note this ring is still Noetherian. Similarly, the completion of $k[x_1, \ldots, x_n]_{\langle f_1, \ldots, f_t \rangle}$ is $k[x_1, \ldots, x_n]_{\langle f_1, \ldots, f_t \rangle}$.

Let us compare the Noetherian ring $(R, m, K)$ with its completion $(\hat{R}, \hat{m}, K)$. Observe that there are natural identifications $R/m^2 \cong \hat{R}/\hat{m}^2$. Intuitively, $R/m^2$ records first order tangent information in a neighborhood of the point $m$. Indeed, this implies that $R$ and $\hat{R}$ have canonically identified Zariski cotangent spaces, hence the same embedding dimensions as well.

Because both the Krull dimension and embedding dimension are unchanged in passing to the completion, we have the following useful fact:

Proposition 3.21. A Noetherian local ring $R$ is regular if and only if its completion $\hat{R}$ is regular.

More generally, the canonical isomorphisms $R/m^t \cong \hat{R}/\hat{m}^t$ imply that Spec $R$ and Spec $\hat{R}$ have the same $t$-th order tangent information for every $t$, so Spec $\hat{R}$ carries all the infinitesimal information of the point $m \in \text{Spec } R$. Indeed, the scheme Spec $\hat{R}$ can be thought of as an infinitesimal neighborhood of the point $m \in \text{Spec } R$.

Example 3.22. Consider $R = k[x, y]/\langle y^2 - x^3 + x \rangle$, the coordinate ring of a plane curve. Complete with respect to the maximal $\langle x, y \rangle$ of the origin. Now, any polynomial in $x, y$ can be rewritten, up through each fixed degree $n$, as a power series only in $y$ (for example, replace all the $x$’s with $x^3 - y^2$, and repeat, until the $x$ degree is too high). It follows that the completion is isomorphic to $k[\hat{y}]$. If you draw a picture of
this variety in the plane, you will see why: infinitesimally near the origin, this variety looks like the $y$-axis.

In particular, even though the completion of $R$ is a power series ring in one variable, the localization $R_{(x,y)}$ is not.

**Example 3.23.** Consider the nodal plane curve defined by the polynomial $y^2 - x^2 - x^3$. Its coordinate ring $R = \mathbb{C}[x, y]/(y^2 - x^2 - x^3)$ can be completed at the maximal ideal $(x, y)$ of the origin to produce the ring $\hat{R} = \mathbb{C}[x, y]/(y^2 - x^2 - x^3)$ whose associated scheme $\text{Spec} \hat{R}$ is an “infinitesimal neighborhood” of the node. In particular, note that the polynomial $y^2 - x^2 - x^3$ factors in the power series ring as $(y - x\sqrt{1 + x})(y + x\sqrt{1 + x})$ where $\sqrt{1 + x}$ is the power series $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{18} - \ldots$, the usual Taylor series expansion for $\sqrt{1 + x}$ at zero. Thus the ring $\hat{R} \cong \mathbb{C}[x, y]/(y - x\sqrt{1 + x})(y + x\sqrt{1 + x})$ “sees” the two branches of the node in a small neighborhood; by contrast, because $R$ is a domain (and remains so after localizing at any element), the two branches can not be observed in any Zariski open set.

In particular, even though $R$ is a domain, its completion $\hat{R}$ need not be.

Complete local rings are in many ways simpler than general Noetherian local rings. The **Cohen Structure Theorem** makes this concrete:

**Theorem 3.24.** (Cohen Structure Theorem, [Mat89, Section 29]) Suppose that $(R, m, k)$ is a complete local Noetherian ring containing any field field. Then $R$ contains a field isomorphic to its residue field and

$$R \cong k[x_1, \ldots, x_n]/I$$

for some ideal $I$. The power series variables $x_i$ can be taken to be minimal generators of the maximal ideal. Furthermore, if $R$ is regular then $R \cong k[x_1, \ldots, x_n]$.

Cohen’s Structure theorem is a non-trivial result that we will take on faith. The hard part of the proof is to show that $R$ contains a field isomorphic to its residue field.

### 3.7. Regularity and Faithfully Flat Extensions.

**Definition 3.25.** A module $M$ over an arbitrary ring $R$ is **faithfully flat** if it is flat and the functor $M \otimes_R -$ is faithful, meaning it takes non-zero $R$-modules to non-zero $R$-modules. To check that a flat module $M$ is faithfully flat, it suffices to check that for every maximal ideal $m$ of $R$, the tensor product $m \otimes_R M \neq 0$.

An $R$-algebra $S$ (equivalently, a ring map $R \to S$) is **faithfully flat** if $S$ is faithfully flat considered as an $R$-module by restriction of scalars.

**Example 3.26.** Let $(R, m, K)$ be a Noetherian local ring. The completion

$$R \to \hat{R}$$

of $R$ at its maximal ideal is faithfully flat [Mat89, Theorem 8.8].
Lemma 3.27. Let \((R, m) \xrightarrow{\phi} (S, n)\) be a local homomorphism of Noetherian local rings. Then \(\phi\) is flat if and only if the induced map on completions \(\hat{R}^m \xrightarrow{\hat{\phi}} \hat{S}^n\) is flat.

Proof. This is a bit tricky to prove but we won’t do it here. See Hochster’s Notes [Hoc07, p54].

In general, regularity descends from faithfully flat extensions:

Theorem 3.28. Let \(R \to S\) be a faithfully flat homomorphism of Noetherian rings. If \(S\) is regular, then \(R\) is regular.

Proof. Let \(P\) be a maximal ideal of \(R\). Then \(PS \neq S\), and there is a maximal ideal \(Q\) of \(S\) lying over \(P\). It suffices to show that every \(R_P\) is regular, and we have that \(R_P \to S_Q\) is flat and local. Thus, we have reduced to the case where \(R\) and \(S\) are local and the map is local. Take a minimal free resolution of \(R/P\) over \(R\). If \(R\) is not regular, this resolution is infinite. Apply \(S \otimes_R \_\). Since \(S\) is \(R\)-flat, we get a free resolution of \(S/PS\) over \(S\). Since \(P\) maps into \(Q\), this resolution is still minimal. Thus, \(S/PS\) has infinite projective dimension over \(S\), contradicting the fact that \(S\) is regular.

4. Kunz’ theorem

Our next goal is to prove the following theorem of Kunz proved in 1969 [Kun69].

Theorem 4.1. If \(R\) is a Noetherian ring of characteristic \(p > 0\), then \(R\) is regular if and only if \(F^* R\) is a flat \(R\)-module.

In the world of varieties, Kunz’s theorem can be stated as follows:

Corollary 4.2. A variety \(X\) over a field of characteristic \(p > 0\) is smooth if and only if the coherent sheaf \(F^* \mathcal{O}_X\) is locally free.

This corollary follows immediately from the fact that a finitely generated flat module over a Noetherian ring is locally free (C. f. Proposition 3.7).

Kunz’s Theorem has the following important consequence:

Corollary 4.3. The regular locus of a Noetherian \(F\)-finite scheme is open. For a Noetherian \(F\)-finite ring, this says that the locus of points \(P \in \text{Spec} R\) where \(R_P\) is regular is open.

Proof. Because openness can be checked on an open cover, the first statement immediately reduces to the second. By Kunz’s theorem, we know that \(R_P\) is regular if and only if \(F^* R_P = (F^* R)_P\) is flat. Since \(F^* R\) is an finitely generated \(R\)-module, this is the equivalent to \((F^* R)_P\) free over \(R_P\) (Proposition 3.7). So the locus of points in \(\text{Spec} R\) such that \(R_P\) is regular is the same as the locus of points in \(\text{Spec} R\) such that \((F^* R)_P\) is free. But the free locus for any coherent sheaf on any Noetherian scheme is open ([Har77][Ex II 5.7]).
Caution: There are Noetherian schemes with non-open regular loci! Hochster gives an example of a Noetherian domain of dimension one which has infinitely maximal ideals \( m \), none of which satisfies \( R_m \) is regular [Hoc]. That is, Hochster’s ring has regular locus consisting only of the zero-ideal, but this is not an open set, since any closed set not containing \((0)\) is finite.

Remark 4.4. This failure of a basic geometric property—openness of the regular locus—in an arbitrary Noetherian scheme frustrated the mid-twentieth century program to generalize all of classical algebraic geometry to arbitrary Noetherian schemes. Indeed, between this and other pathologies that were discovered, Grothendieck introduced the notion of an excellent ring, which is basically a list of axioms designed to rule out such pathologies. It is still a dream of many today that “all of algebraic geometry”—for example, resolutions of singularities—should be feasible for excellent (or perhaps locally excellent) schemes.

The condition of F-finiteness for a Noetherian scheme is fairly mild and implies the important geometric property that the regular locus is open. In fact, it turns out that F-finiteness implies all the good properties envisioned by Grothendieck as the minimal setting for algebraic geometry. That is, an F-finite Noetherian scheme is excellent [Kun76]. We will not go further into excellence in this course.

Observe that if \( R \to F_* R \) is flat, then the composition \( R \to F^e_* R \) (of the Frobenius with itself \( e \) times) for every \( e \) is flat as well. In fact, we will prove the slightly stronger statement of Kunz:

**Theorem 4.5.** If \( R \) is a Noetherian ring of characteristic \( p > 0 \), then \( R \) is regular if and only if \( F^e_* R \) is a flat \( R \)-module for some (equivalently, every) \( e \in \mathbb{N} \).

**Remark 4.6.** If \( F_* R \) is flat over \( R \), then it is obviously faithfully flat. Indeed, for any maximal ideal \( m \), the module \( m \otimes_R F_* R = mF_* R = F_* (m^{[p^e]}) \neq F_* R \). The same obviously holds for for \( F^e_* R \) as well.

Our strategy for proving Theorem 4.5 is to reduce to the complete local case, where the Cohen-Structure theorem makes the ring more concrete. Because both flatness and regularity are local, the proof reduces immediately to the local case. We also have seen that a local Noetherian ring \( R \) is regular if and only if its completion is regular. Fortunately, the Frobenius map plays well with respect to completion, too.

9Contrary to what you might see posted by strong mathematicians on Math Overflow! Don’t believe everything you read!
4.1. Frobenius and Completion. Let \((R, m)\) be a local ring. Consider the following two compositions
\[ R \to F_* R \to (F_* R) \]
\[ R \to \hat{R} \to F_* \hat{R}. \]
In the first, we follow Frobenius \(R \to F_* R\) by the natural completion map at \(m\). In the second, we first complete, then apply Frobenius to \(\hat{R}\). Fortunately, for Noetherian rings, these two compositions are essentially the same: Frobenius commutes with completion.

**Lemma 4.7.** Let \(R\) be a Noetherian local ring, and let \(\hat{R}\) denote its completion at the unique maximal ideal. Then we have a canonical identification of the maps
\[ \hat{R} \to (F_* R) \quad \text{and} \quad \hat{R} \to F_* \hat{R} \]
where the first map is the completion of the Frobenius map for \(R\) and the second map is the Frobenius on \(\hat{R}\). Likewise, the same statement holds for any iterate \(F^e\) of Frobenius.

**Proof.** Let \(m\) denote the maximal ideal of \(R\). By definition, \((F_* R)\) is \(\lim_{\leftarrow} F_* R/m^n F_* R\) and the map
\[ \hat{R} \to (F_* R) \]
is obtained by taking the inverse limit of the natural maps \(R/m^n \to (F_* R)/m^n (F_* R)\).

There are natural isomorphisms
\[ F_* R/m^n F_* R \cong F_* R/F_* (m^n)^{[p]} \cong F_* (R/(m^n)^{[p]}) \]
for all \(n\).

We claim that \(\{(m^n)^{[p]}\}_n\) and \(\{m^n\}_n\) are cofinal with each other as we range over all \(n\). Indeed, since \((m^n)^{[p]} \subset m^n\) for all \(n\), we only need to check that for each \(n\), there is some \(N\) such that \(m^N \subset (m^n)^{[p]}\). We claim that \(N = pdn\) works, where \(d\) is the number of generators for \(m\). To prove this, consider \(m^{pd} = \langle x_1, x_2, \ldots, x_d \rangle^{pd}\). It is generated by monomials \(x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d}\) of degree \(pd\) in the generators of \(m\). Note that \(a_i \geq p\) for at least one \(a_i\) in each such monomial. So \(m^{pd} \subset m^{[p]}\), and hence \(m^{pdn} \subset (m^{[p]})^n\) for all \(n\).

Because \(\{(m^n)^{[p]}\}_n\) and \(\{m^n\}_n\) are cofinal, they define the same inverse limits. That is, \(\lim_{\leftarrow} F_* R/m^n F_* R \cong F_* \hat{R}\). Thus the completion of the Frobenius map \(R \to F_* R\) produces a natural map
\[ \hat{R} \to \lim_{\leftarrow} F_* (R/(m^n)^{[p]}) \cong F_* \hat{R} \]
which is the \(p\)-th power map on \(\hat{R}\). \(\square\)

**Remark 4.8.** If \(R\) is \(F\)-finite, then also \(\hat{R} \otimes_R F_* R \cong F_* \hat{R}\), so we can also view the Frobenius on \(\hat{R}\) as obtained from \(R \to F_* R\) by tensoring with \(\hat{R}\). However, in general for an infinitely generated \(M\), it is not the case that \(\hat{M} \cong R \otimes_R M\).
Proposition 4.9. Let \((R, m)\) be a Noetherian local ring of prime characteristic \(p\). Then the Frobenius map \(R \to F_* R\) is flat if and only if the Frobenius map \(\hat{R} \to F_* \hat{R}\) is flat. Likewise, the same holds for the composition \(F^e\) of Frobenius with itself \(e\) times, for any \(e \in \mathbb{N}\).

**Proof.** Assume first that \(F_* R\) is flat over \(R\). Then \(\hat{F}_* \hat{R}\) is flat over \(\hat{R}\). By Lemma 4.7, this means \(F_* \hat{R}\) is flat over \(\hat{R}\).

Conversely, assume that \(F_* \hat{R}\) is flat over \(\hat{R}\). Consider the following diagram:

\[
\begin{array}{ccc}
\hat{R} & \xrightarrow{F} & F_* \hat{R} \\
\uparrow & & \uparrow \\
R & \xrightarrow{F} & F_* R
\end{array}
\]

The vertical arrows, both being completion, are faithfully flat (note that right vertical arrow is essentially just a renaming of the left arrow). Because we are assuming that the upper arrow is flat, and a composition of flat maps is flat, we know that the composition \(R \to F_* R \to F_* \hat{R}\) is flat.

The bottom horizontal arrow \(R \to F_* R\) is flat by an immediate application of the following simple lemma:

**Lemma 4.10.** If \(A \to B \to C\) is flat and \(B \to C\) is faithfully flat, then \(A \to B\) is flat.

**Proof of Lemma.** Say \(M' \to M\) is an injection of \(A\) modules. Let \(K\) be the kernel of \(M' \otimes_A B \to M \otimes_A B\). We need to show that \(K = 0\). We have an exact sequence \(0 \to K \to M' \otimes_A B \to M \otimes_A B\) of \(B\)-modules. We tensor this over \(B\) with \(C\) to obtain

\[
\begin{array}{ccc}
0 & \longrightarrow & K \otimes_B C \\
\downarrow & & \downarrow \\
0 & \longrightarrow & M' \otimes_A B \otimes_B C \\
\downarrow & & \downarrow \\
0 & \longrightarrow & M \otimes_A B \otimes_B C
\end{array}
\]

Since \(B \to C\) is faithfully flat, \(K \otimes_B C \neq 0\) if \(K \neq 0\). But \(M' \otimes_A C \to M \otimes_A C\) must be injective by the flatness of \(C\) over \(A\). This completes the proof of the lemma as well as the proof of Proposition 4.9. □

**Remark 4.11.** Proposition 4.9 also follows immediately from Lemma 3.27 and Proposition 4.7, but we wanted to give a direct proof in this case.

\[^{10}\text{See Lemma 15.27.4 in the Stacks Project. Sta16 Tag 06LD}\]
4.2. The proof of Kunz’s Theorem. Let \( R \) be an arbitrary Noetherian ring of prime characteristic. By definition, \( R \) is regular if and only if \( R_m \) is regular for every maximal ideal of \( R \). But also for every \( e \in \mathbb{N} \), the module \( F^e R \) is flat over \( R \) if and only if the localization \( (F^e R)_m \) is flat over \( R_m \) for all maximal ideals (Proposition 3.6). Since Frobenius commutes with localization (Proposition 1.15), we know \( (F^e R)_m \cong F^e R_m \). So \( F^e R \) is flat over \( R \) if and only if \( F^e R_m \) is flat over \( R_m \) for all maximal ideals \( m \) of \( R \). Therefore the proof of Kunz’s Theorem reduces to the local case.

Now consider the local Noetherian ring \((R, m, K)\). We know that \( R \) is regular if and only if \( \hat{R} \) is regular, and also that Frobenius is flat on \( R \) if and only if Frobenius is flat on \( \hat{R} \) (Proposition 4.9). So we have also reduced to the complete local case.

4.2.1. Proof that Regular implies Frobenius is flat: Let \((R, m, K)\) be a complete regular local ring of prime characteristic. By the Cohen Structure Theorem, we can write \( R = \hat{K}[x_1, \ldots, x_n]/I \) where \( I \) is some ideal of the power series ring \( \hat{K}[x_1, \ldots, x_n] \). Here, \( n \) can be assumed the embedding dimension of \( R \).

Step 1: We first introduce Lech Independence and prove Lemma 4.13 stating that powers of the generators of the maximal ideal of \( R \) are Lech independent under our assumption that \( F^e \) is flat.

Definition 4.12. A sequence of elements \( f_1, \ldots, f_s \) of an arbitrary ring is called **Lech independent** if whenever \( a_1 f_1 + \ldots + a_s f_s = 0 \) for some elements \( a_i \in R \), then each \( a_i \in \langle f_1, \ldots, f_s \rangle \).

For example, any minimal set of generators for the maximal ideal of a Noetherian local ring is Lech Independent. When Frobenius is flat, then any powers of the minimal generators are too:
Lemma 4.13. Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of prime characteristic $p$, and let $x_1, \ldots, x_n$ be a minimal set of generators for $\mathfrak{m}$. If some iterate of Frobenius on $R$ is flat, then the set

$$\{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\}$$

is Lech independent for any positive $\alpha_1, \ldots, \alpha_n$.

Proof. Assume that $F^e$ is flat. We will prove Lemma 4.13 in two steps. Our first step is to show that for each $t \in \mathbb{N}$, the set $\{x_1^{p^t}, \ldots, x_n^{p^t}\}$ is Lech independent for all $t \in \mathbb{N}$. This follows\footnote{The exposition here has been enhanced by suggestions of Yifeng Huang and Hang Yin, who each suggested different simplifications of the argument.} from a more general result for any flat map: If $f_1, \ldots, f_n$ are elements of arbitrary ring $R$ generating an ideal $J$ with the property that $J/J^2$ is free of rank $n$ over $R/J$, and then the images $\phi(f_1), \ldots, \phi(f_n)$ under any flat map $R \to S$ are Lech independent in $S$. To prove this, observe that since $J/J^2$ is free of rank $n$ over $R/J$, then tensoring with the flat algebra $S$, also $JS/(JS)^2$ is free over $S/JS$ of rank $n$. Now any relation $a_1\phi(f_1) + \cdots + a_n\phi(f_n) = 0$ induces a relation on the free generators $\phi(f_i)$ (really their classes $\phi(f_i)$ modulo $(JS)^2$) of $JS/(JS)^2$, which forces each $a_i \in (JS)$. That is, the images of the $f_i$ under the flat map $\phi$ form a Lech independent set.

Returning to our situation at hand, let $\{x_1, \ldots, x_n\}$ be a minimal generating set for $\mathfrak{m}$. Because $\mathfrak{m}/\mathfrak{m}^2$ is free of rank $n$ over $R/\mathfrak{m}$, the images $\{x_1^{p^t}, \ldots, x_n^{p^t}\}$ under the flat map $F^e$ (and under the flat map $F^{e^t}$ obtained by composing $F^e$ with itself $t$ times) are Lech independent. That is, the set $\{x_1^{p^t}, \ldots, x_n^{p^t}\}$ is Lech Independent for all $t \in \mathbb{N}$.

The second and final step in the proof of Lemma 4.13 uses the next lemma to inductively reduce the exponents on the Lech independent set $\{x_1^{p^t}, \ldots, x_n^{p^t}\}$ down to $\{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}\}$.

Lemma 4.14. Let $f_1, \ldots, f_s$ be Lech independent elements of an arbitrary ring, and assume that $g_1$ divides $f_1$. Then $g_1, f_2, \ldots, f_s$ is also Lech independent.

Proof. Write $f_1 = g_1h_1$. Suppose $a_1g_1 + a_2f_2 + \cdots + a_sf_s = 0$. Then multiplying by $h_1$, we have $a_1f_1 + (a_2h_1)f_2 + \cdots + (a_sh_1)f_s = 0$ so by definition of Lech independent, we have that $a_1 \in \langle f_1, \ldots, f_s \rangle$ and that $a_ih_1 \in \langle f_1, \ldots, f_s \rangle$ for all $i \geq 2$. This implies immediately that $a_1 \in \langle g_1, f_2, \ldots, f_s \rangle$. To see that also $a_i \in \langle f_1, \ldots, f_s \rangle$ for $i \geq 2$, write

$$a_1 = b_1f_1 + b_2f_2 + \cdots + b_sf_s$$

and substitute to get

$$(b_1f_1 + b_2f_2 + \cdots + b_sf_s)g_1 + a_2f_2 + \cdots + a_sf_s = 0.$$
This means that each \( g_ib_i + a_i \) is in \( \langle f_1, \ldots, f_s \rangle \), so that \( a_i \in \langle g_1, \ldots, f_s \rangle \). This completes the proof of the Lemma. \( \square \)

Note that Lemma [4.13] is now proved as follows: Choose \( t \) so that \( p^t \geq \alpha_i \) for all \( i \). Then repeated applications of Lemma [4.14] reduce the exponent of each \( x_i \) down to \( \alpha_i \). \( \square \)

**Remark 4.15.** The first three sentences of the proof of Lemma [4.14] also show that if \( g_1 \) divides \( f_1 \) (and \( \{f_1, \ldots, f_s\} \in R \) is Lech independent) then \( \langle f_2, \ldots, f_s \rangle : g_1 \subseteq \langle f_1, \ldots, f_s \rangle \). Recall the notation: by definition \( I : g = \{r \in R \mid gr \in I\} \).

**Step 2:** We use Lech independence to prove that the quotient ring \( R/\langle m^{[p^t]} \rangle \) has dimension \( p^{nt} \) over \( K \) for all \( t \), where \( n \) is the embedding dimension of \( R \). For this we need the following general lemma:

**Lemma 4.16.** [Lec64] **Lemma 4** Let \( \{f_1, \ldots, f_s\} \) be a Lech independent set of an arbitrary ring containing a field \( K \), and suppose that \( f_1 = g_1h_1 \). If \( \dim_K \left( R/\langle f_1, \ldots, f_s \rangle \right) \) is finite, then

\[
\dim_K \left( R/\langle f_1, \ldots, f_s \rangle \right) = \dim_K \left( R/\langle g_1, f_2, \ldots, f_s \rangle \right) + \dim_K \left( R/\langle h_1, f_2, \ldots, f_s \rangle \right)
\]

**Proof.** The lemma follows immediately from the following short exact sequence

\[
0 \rightarrow R/\langle h_1, f_2, \ldots, f_s \rangle \xrightarrow{g_1} R/\langle f_1, \ldots, f_s \rangle \rightarrow R/\langle g_1, f_2, \ldots, f_s \rangle \rightarrow 0
\]

Here, the non-trivial mapping on the right is the natural quotient map, and the non-trivial mapping on the left is multiplication by \( g_1 \), which is injective by Remark [4.15]. Indeed, if the class of \( r \) is in its kernel, we have \( rg_1 \in \langle f_1, \ldots, f_s \rangle \), so that we can write \( rg_1 = a_1g_1h_1 + a_2f_2 + \cdots + a_sf_s \). Rearranging, we have \( g_1(r - a_1h_1) \in \langle f_1, \ldots, f_s \rangle \), so that using Remark [4.15] we have \( r - a_1h_1 \in \langle f_1, f_2, \ldots, f_s \rangle \subseteq \langle h_1, f_2, \ldots, f_s \rangle \), whence \( r \in \langle h_1, f_2, \ldots, f_s \rangle \), as needed. \( \square \)

To prove the claim in Step 2, we apply the lemma to our complete local ring \( (R, m, K) \) to show that

\[
\dim_K \frac{R}{\langle x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} \rangle} = \alpha_1 \cdot \alpha_2 \cdots \alpha_n.
\]

This follows immediately by induction on \( \alpha_1 + \cdots + \alpha_n \), noting that in the base case when \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = 1 \), the dimension of \( R/m = 1 \). In particular, the claim follows: \( R/\langle m^{[p^t]} \rangle \) has dimension \( p^{nt} \) over \( K \).

**Step 3:** Finally, we complete the proof that if some iterate of Frobenius is flat for the complete local ring \( (R, m, K) \), then \( R \) is regular. By the Cohen Structure Theorem, we can write

\[
R \cong K[x_1, \ldots, x_n]/I
\]

for some ideal of \( I \) of \( K[x_1, \ldots, x_n] \). We know that

\[
\dim_K K[x_1, \ldots, x_n]/\langle x_1^{p^t}, \ldots, x_n^{p^t} \rangle = p^{tn}
\]
for all \( t \) by direct computation (a \( K \)-basis consists of monomials \( x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \) where \( 0 \leq a_i < p \)). We also know that \( R/\langle x_1^{p_1}, \ldots, x_n^{p_n} \rangle \) has dimension \((p^n)^n = p^{tn}\) for all \( t \) from Step 2. But then the quotient map

\[
\frac{K[x_1, \ldots, x_n]}{\langle x_1^{p_1}, \ldots, x_n^{p_n} \rangle} \to \frac{K[x_1, \ldots, x_n]}{I + \langle x_1^{p_1}, \ldots, x_n^{p_n} \rangle} \cong R/m^{[p^t]}
\]

must be an isomorphism for every \( t \). This means that \( I \subset m^{[p^t]} \) for all \( t \), so that \( I \subset \cap_t m^{p^t} = 0 \). We conclude that \( R = K[x_1, \ldots, x_n] \), so that \( R \) is regular. This completes the proof of Kunz’s Theorem.

**Remark 4.17** (Hilbert-Kunz Multiplicity). For a local ring \((R, m, K)\), the Hilbert-Kunz multiplicity is defined to be the limit:

\[
e_{HK}(R) = \lim_{e \to \infty} \dim_K \left( R/m^{[p^e]} \right).
\]

Notice that if \( F_*R \) is a flat \( R \)-module (that is, when \( R \) is regular), Step 3 above shows that \( e_{HK}(R) = 1 \). We will see later that this limit exists in general, and indeed, is equal to 1 if and only if \( R \) is regular.

### 4.3. Other Proofs of Kunz’s Theorem.

There are several different proofs of Kunz’s theorem. One of particular interest is a recent proof of Bhatt and Scholze, which uses the perfection of a ring\(^{\text{12}}\).

Let us sketch the main ideas of the Bhatt-Scholze argument. Zhan Jiang has promised to give us a talk which will flesh these ideas out.

**Definition 4.18.** Let \( R \) be a ring of prime characteristic \( p \). We say that \( R \) is **perfect** if the Frobenius map \( R \overset{F}{\to} R \) is an isomorphism.

Because Frobenius is injective if and only if \( R \) is reduced, we can also define a perfect ring as a reduced ring in which every element has a \( p \)-th root. This terminology agrees with the usual terminology for perfect fields. Indeed, a field of characteristic \( p \) is defined to be **perfect** in most elementary algebra classes when \( K^p = K \). In particular, every perfect field, including all finite fields and all algebraically closed fields, are perfect rings.

**Example 4.19.** The ring \( \mathbb{F}_p[x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, \ldots] \) obtained by adding all \( p^e - th \) roots to \( \mathbb{F}_p[x] \) is perfect.

Every ring of characteristic \( p \) admits a natural map to a perfect ring:

**Definition 4.20.** Let \( R \) be a ring of prime characteristic \( p \). The **perfection** of \( R \) is the direct limit ring

\[
R^\infty = \lim_{\to F} R = \lim_{\to F} (R \overset{F}{\to} R \overset{F}{\to} R \overset{F}{\to} R \cdots)
\]

where the transition maps are Frobenius.

\(^{\text{12}}\)Other approaches can be found in [Her74] and [MR10, Theorem 4.4.2].
Remark 4.21. The perfection of $R$ is obviously a perfect ring. There is a natural ring homomorphism $R \to R^\infty$. We can alternatively define a perfect ring as one such that the natural map $R \to R^\infty$ is an isomorphism.

If $R$ is reduced (for example, a domain), we can identify the perfection with the ring $R^{1/p^\infty} = \bigcup_{e \geq 0} R^{1/p^e}$.

Example 4.22. The perfection of the ring $R = \mathbb{F}_p[[x]]$ is the perfect ring $R^\infty = \mathbb{F}_p[x, x^{1/p}, x^{1/p^2}, x^{1/p^3}, \ldots]$.

Remark 4.23. Note that the perfection of $R$ is rarely Noetherian, even when $R$ is. Indeed, the only Noetherian perfect rings are finite products of perfect fields. On the other hand, it is easy to check that Spec $R^\infty \to$ Spec $R$ is a homeomorphism of the underlying topological spaces. In particular, when $R$ is a Noetherian local ring, its perfection has finite Krull dimension.

Remark 4.24. The perfection $R^\infty$ is always reduced even when $R$ is not. Indeed, the Frobenius map on $R^\infty$ is always injective: if something is killed by Frobenius, it is also killed by some map defining the direct limit.

It is easy to see that for any map of rings $R \to S$, there is an induced compatible map of their perfections $R^\infty \to S^\infty$. Similarly, if $R$ is an arbitrary map that admits a homomorphism to a perfect ring $R \to S^\infty$, then this map must factor uniquely through the perfection of $R$:

$$R \to R^\infty \to S^\infty.$$ 

Remark 4.25. There is another natural way to get a map from an arbitrary ring $R$ of characteristic $p$ to a perfect ring—namely, we get take the inverse limit of the Frobenius maps instead:

$$R^\flat = \lim_{\leftarrow F} R = \lim_{\leftarrow F}(R \leftarrow^F R \leftarrow^F R \leftarrow^F \cdots).$$

The perfect ring $R^\flat$ is called the tilting of $R$. For example, if $R = \mathbb{F}_p[[x]]$, then $R^\flat \cong \mathbb{F}_p$. In the tilt, rather than add in $p^e$-th roots of elements like we did to get the perfection, we throw away all elements that do not have $p^e$-th roots for all $e$. We will not need this construction here.

Bhatt and Scholze prove the following beautiful theorem:

Theorem 4.26. [BS15 Proposition 11.31] Let $R^\infty$ be the perfection of a complete local ring $R$. Then $R^\infty$ has finite global dimension.
A ring has **finite global dimension** if there is an upper bound on the projective dimension for all $R$-modules. Serre’s theorem says that a Noetherian local ring has finite global dimension if and only if it is regular. The Bhatt-Scholze theorem says that perfect rings, while not Noetherian, share this property with regular local rings.

Theorem 4.26 is a non-trivial result; see [https://rankeya.people.uic.edu/Kunz’s theorem perfect rings.pdf](https://rankeya.people.uic.edu/Kunz’s theorem perfect rings.pdf) for a nice exposition. It yields the harder direction of Kunz’s theorem as an immediate corollary.

**Corollary 4.27.** Suppose $R$ is a Noetherian ring such that $F_* R$ is a flat $R$-module. Then $R$ is regular.

**Proof.** We have already explained how the proof reduces to the complete local case. In this case, it suffices to check that $R$ has finite global dimension. For this, we need to check that for any finitely generated $R$-module $M$, $\text{Tor}_n^R(M, K) = 0$ for $n \gg 0$. Because a direct limit of flat maps is flat, we see that $R^\infty$ is a flat $R$-module; moreover, it is *faithfully* flat since $m R^\infty \neq R^\infty$. Thus it suffices to check that $R^\infty \otimes_R \text{Tor}_n^R(M, K) = 0$ for $n \gg 0$. By flatness of $R^\infty$, this is the same as checking that $\text{Tor}_n^{R^\infty}(R^\infty \otimes_R M, R^\infty \otimes_R K) = 0$. But this is immediate from the theorem of Bhatt and Scholze: every module over $R^\infty$ has a finite projective resolution from which we can compute its Tor.

Now that $R$ has finite global dimension, we conclude that $R$ is regular as claimed. \qed
CHAPTER 3

Frobenius splitting and F-regularity

We have seen that the Frobenius map can be used to identify when a Noetherian ring \( R \) (or variety \( X \)) of prime characteristic is regular (respectively, smooth). Indeed, Kunz’s theorem says that \( R \) is regular if and only if Frobenius is flat. For varieties, this amounts to saying \( X \) is a smooth if and only if the coherent sheaf \( F_*\mathcal{O}_X \) is locally free.

Our next goal is to relax the assumption that \( R \) is flat while still maintaining some nice properties. By doing so, we introduce classes of “F-singularities” that, as we will see, have very nice properties.

In this chapter, we first introduce F-singularities called Frobenius splitting and F-regularity, and prove their basic properties in both local and global settings. We compare global Frobenius splitting for projective varieties with local Frobenius splitting at the “vertex of the cone” in Section 7. We establish vanishing theorems for cohomology of certain sheaves using Frobenius splitting in Section §5.1. Finally, we develop techniques for identifying Frobenius split schemes in Section §4, including Fedder’s Criterion for Frobenius splitting of a variety in terms of its defining equations. Ultimately, we aim to prove that F-regular singularities correspond in a suitable sense to what is known as Kawamata log terminal singularities in the minimal model program, and Frobenius split singularities (conjecturally) to log canonical singularities.

1. Frobenius Splitting

Definition 1.1. Let \( R \) be a ring of characteristic \( p \). We say that \( R \) is Frobenius split if there exists an \( R \)-module map \( F_*R \xrightarrow{\pi} R \) that splits the Frobenius map \( R \xrightarrow{F} F_*R \) in the category of \( R \)-modules—that is, such that the composition \( \pi \circ F \) is the identity map of \( R \).

Note that finding a Frobenius splitting for a ring \( R \) amounts to finding an \( R \)-module homomorphism \( \phi \in \text{Hom}_R(F_*R, R) \) such that \( \phi(F, 1) = 1 \). Indeed, in this case, for every \( r \in R \), we have \( \phi(F(r)) = \phi(rF_*(1)) = r\phi(F_*(1)) = r \), so any such \( \phi \) gives a splitting of the Frobenius map.

Note also that \( R \) is Frobenius split if and only if the \( R \)-module \( F_*R \) can be decomposed as a direct sum \( F_*R \cong R \oplus M \) where \( M \) is some other \( R \)-module (in fact \( M \cong \ker \phi \) where \( \phi \in \text{Hom}_R(F_*R, R) \) as in the preceding paragraph).
Frobenius splitting of a local ring can be interpreted as a kind of restriction on the singularity at the corresponding point. For example, Frobenius splitting already rules out the worst pathologies:

**Proposition 1.2.** A Frobenius split ring is reduced.

**Proof.** Let $R$ be a Frobenius split ring. Because the composition 

$$ R \xrightarrow{F} F_* R \xrightarrow{\pi} R $$

is the identity map, in particular $F$ must be injective. As we have proved (Proposition 1.5), this means that $R$ is reduced. \qed

You should think of splitting of Frobenius as a weakening of flatness of Frobenius. That is, the least singular rings are always Frobenius split:

**Proposition 1.3.** Every F-finite regular local ring is Frobenius split. In particular, the local ring of a smooth point on a variety of characteristic $p$ is Frobenius split.

**Proof.** Since varieties are always F-finite and smooth points on varieties are always regular, the second sentence is immediately implied by the first.

So suppose that $(R, m)$ is an F-finite regular local ring. Then $F_* R$ is both finitely generated and flat over the local ring $R$, hence free. Any minimal set of generators will be a free basis; we can find one by choosing a basis for $F_* R / m F_* R = F_* (R/ m^{[p]})$. Clearly, since $1 \notin m^{[p]}$, we can take $F_* 1$ to be among a free basis for $F_* R$ over $R$. The projection onto the $R$-submodule spanned by $F_* 1$ is an $R$-linear map 

$$ \phi : F_* R \to R $$

which sends the free basis element $F_* 1$ to 1 and all other basis elements to zero. It is easy to verify that $\phi \circ F$ is the identity map on $R$. Thus $R$ is Frobenius split. \qed

As usual, we can extend the definition of Frobenius splitting to the case of schemes:

**Definition 1.4.** Let $X$ be an arbitrary scheme of prime characteristic $p$. We say that $X$ is Frobenius split if the Frobenius map $\mathcal{O}_X \to F_* \mathcal{O}_X$ splits in the category of $\mathcal{O}_X$-modules. Explicitly, this means that there is a $\mathcal{O}_X$-module map $\pi : F_* \mathcal{O}_X \to \mathcal{O}_X$ such that the composition

$$ \mathcal{O}_X \xrightarrow{F_*} F_* \mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_X $$

is the identity map of the sheaf $\mathcal{O}_X$.

In particular, the scheme $\text{Spec } R$ is Frobenius split if and only if the ring $R$ is Frobenius split.

Caution is in order, however: splitting a map of sheaves on a non-affine scheme is not a local condition! We can not check whether or not the Frobenius map $\mathcal{O}_X \xrightarrow{F_*} F_* \mathcal{O}_X$ on an arbitrary scheme splits by checking it on an affine cover, nor by checking for splitting at each stalk: Frobenius splitting is a global property which has to do with whether or not we can glue together local splittings of Frobenius.
In particular, our straightforward-seeming extension of the definition of Frobenius splitting is no longer a weakening of the flatness of Frobenius! Flatness of Frobenius is a local condition. We are therefore motivated to introduce a different way to extend Definition 1.1 to an arbitrary scheme:

**Definition 1.5.** Let \( X \) be a scheme of characteristic \( p \). We say that \( X \) is **Frobenius split** at \( x \in X \) if the local ring \( O_{X,x} \) (the stalk at \( x \)) is Frobenius split. We say \( X \) is **locally Frobenius split** if it is Frobenius split at every point.

It is easy to check that a Frobenius split scheme is always locally Frobenius split, or similarly, that the localization of a Frobenius split ring at any multiplicative set remains Frobenius split. However, the local condition is strictly weaker than the global condition in general!

An F-finite regular scheme is always locally Frobenius split by Proposition 1.3. Likewise, a locally Frobenius split scheme is always reduced by Proposition 1.2.

To avoid confusing the local and global types of Frobenius splitting, we will sometimes emphasize that a scheme \( X \) is **globally Frobenius split** in referring to Definition 1.4. Both global Frobenius splitting and local Frobenius splitting are powerful tools in algebraic geometry, but they are good for different things: global Frobenius splitting can inform us about the global geometry of a projective variety (genus, etc) whereas local Frobenius splitting is a restriction on the singularities. We will return to global Frobenius splitting in Section 5.

**Remark 1.6.** The assumption that \( R \) is **F-finite**—that is, that the Frobenius map is finite (See Definition 1.17)— in Proposition 1.3 is necessary, but fortunately it is a mild condition that will be satisfied in the rings of interest in this course. For example, recall that finitely generated algebras over a perfect (or \( F \)-finite) field are always F-finite, as are their localizations and completions. Frobenius splitting is best behaved in the F-finite setting, where the proofs are more elegant and more can be done. See also Remark 1.7.

**Remark 1.7.** For a non-F-finite ring, even a regular one, it can happen that the module \( \text{Hom}_R(F_*R, R) \) is the zero module! Clearly in this case, \( R \) has no hope to be Frobenius split. (See [DS18] for an example where this happens for a DVR.) Hochster and Roberts introduced **F-purity**, which is equivalent to Frobenius splitting in the F-finite case yet more useful in the non-F-finite case [HR74], and proved all regular rings of characteristic \( p \) are F-pure. We will not go into this now, since we are primarily interested in F-finite rings. We may return to this if we have time later.
2. First Examples of locally Frobenius Split Varieties

Before continuing with proving properties of Frobenius splitting, we present a few examples and non-examples.

**Example 2.1.** The polynomial ring \( R = \mathbb{F}_p[x_1, \ldots, x_d] \) is Frobenius split. Indeed, we have proved that \( F_* R \) is free on the basis \( \{ x_1^{a_1} \cdots x_d^{a_d} \} \) where \( 0 \leq a_i < p \). Among these free basis elements is the element \( 1 = x_1^0 \cdots x_d^0 \). Projection onto the summand spanned by 1 gives an \( R \)-linear map \( F_* R \to R \) which sends the basis element 1 to 1, and all other basis elements to 0. This is a splitting of the Frobenius map \( R \hookrightarrow F_* R \), whose image is exactly the \( R \)-span of 1.

Indeed, the Frobenius map splits for any polynomial ring of characteristic \( p \) regardless of the ground field. This is easy to see by factoring the Frobenius map as the composition of split inclusions \( K[x_1, \ldots, x_d] \hookrightarrow K[x_1^{1/p}, \ldots, x_d^{1/p}] \hookrightarrow K^{1/p} \otimes_K K[x_1^{1/p}, \ldots, x_d^{1/p}] = K^{1/p}[x_1^{1/p}, \ldots, x_d^{1/p}] \).

The first map is free on the usual basis, and the second map is free on the basis \( \nu_i \otimes 1 \) where \( \{ \nu_i \} \) is a basis for \( K^{1/p} \) over \( K \). Thus \( K^{1/p}[x_1^{1/p}, \ldots, x_d^{1/p}] \) is free over \( K[x_1, \ldots, x_d] \) on the basis \( \{ \nu_i x_1^{a_1} \cdots x_d^{a_d} \} \). In particular, we can project onto the summand spanned by \( 1 \cdot x_1^{a_1} \cdots x_d^{a_d} \) to get a splitting of \( R \subset R^{1/p} \).

**Example 2.2.** The variety \( \mathbb{P}^n_k \) where \( k \) is an arbitrary field of prime characteristic is locally Frobenius split (also globally Frobenius split; see Example 7.1.3). This follows from the previous example since every local ring of \( \mathbb{P}^n_k \) is the localization of a polynomial ring.

**Exercise 2.1.** Show that \( K[x_1, \ldots, x_d] \) is Frobenius split, where \( K \) is an arbitrary field of characteristic \( p \). [Hint: Consider completing an appropriate map \( \phi \in \text{Hom}_R(F_* R, R) \) at the maximal ideal; see Lemma 4.7]

**Example 2.3.** Any regular scheme of finite type over a perfect field \( k \) of characteristic \( p \) is locally Frobenius split. For example, smooth varieties of prime characteristic are always locally Frobenius split (but rarely globally Frobenius split; see Section 5).

The next proposition is useful for finding non-regular examples of Frobenius split rings, including many rings of invariants:

**Proposition 2.4.** Let \( S \to R \) be any homomorphism of rings which splits as a map of \( S \)-modules. If \( R \) is Frobenius split, then so is \( S \).

**Proof.** The hypothesis means that there is an \( S \)-linear map \( R \to S \) sending \( 1_R \to 1_S \). Assuming that \( R \) is Frobenius split, we fix a splitting let \( F_* R \to R \). Consider the composition

\[
S \to F_* S \xrightarrow{F_* \phi} F_* R \xrightarrow{\pi} R \xrightarrow{\phi} S
\]
where the first arrow is the Frobenius on \( S \). All these maps are \( S \)-module maps (since any \( R \)-linear map is automatically linear over the subring \( S \)), and one easily checks that the composition is the identity map on \( S \). This means that \( S \to F_* S \) is split by the composition \( \phi \circ \pi \circ F_* \iota \). So \( S \) is Frobenius split. \( \square \)

**Example 2.5.** Proposition 2.4 provides a host of examples of Frobenius split rings contained in a polynomial ring. For example, any Veronese subring \( K[x_1, \ldots, x_d]^{(n)} := K[\{\text{monomials of degree } n\}] \subset K[x_1, \ldots, x_d] \)
is Frobenius split, since Veronese subrings are direct summands of the polynomial ring (simply by mapping any homogenous element whose degree is a multiple of \( n \) to itself and killing all others).

**Example 2.6.** Likewise, the ring of invariants for any finite group \( G \) acting on a polynomial ring \( R \) is Frobenius split, provided \( p \) does not divide \( |G| \). This is because we have a splitting of \( R^G \hookrightarrow R \) defined by “averaging the orbit” of each element \( r \):

\[
R \to R^G \quad r \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot r
\]

whenever \( |G| \neq 0 \) in \( k \).

Neither Veronese subrings nor rings of invariants are typically regular, so the previous Examples provide *non-regular* examples of rings (or non-smooth varieties) that are locally Frobenius split.

**Example 2.7.** For a non-Frobenius split example, consider the ring

\[
R = \mathbb{F}_p[x, y, z]/(x^7 + y^7 - z^7)
\]

where the prime \( p \neq 7 \). The ring \( R \) is a free module over its subring \( A = \mathbb{F}_p[x, y] \) on the basis \( 1, z, \ldots, z^6 \). In particular, note that \( z^6 \notin (x, y)R \). Note however that \( (z^6)^p \) has degree \( 6p \), so when we write it as a homogeneous \( A \)-linear combination of these free basis elements,

\[
a_0 + a_1 z + \cdots + a_6 z^6,
\]

the coefficients \( a_i \) will be polynomials in \( \mathbb{F}_p[x, y] \) of degrees \( 6p, 6p - 1, \ldots, 6p - 6 \), respectively. The pigeon hole principle implies that any monomial in \( x \) and \( y \) of degree at least \( 2p - 1 \) must be in \((x^p, y^p)\); in particular, since \( 6p - 6 > 2p - 2 \), all coefficients \( a_i \in (x^p, y^p) \) so that \( z^{6p} \in (x^p, y^p)R \).

Now suppose that \( R \) is Frobenius split and let \( \phi : F_* R \to R \) be a splitting of Frobenius. We can apply \( \phi \) to the expression \( z^{6p} = r_1 x^p + r_2 y^p \) to get

\[
\phi(F_*(z^{6p})) = \phi(F_*(r_1 x^p)) + \phi(F_*(r_2 y^p))
\]

\[
z^6 \phi(F_*)1 = x\phi(F_* r_1) + y\phi(F_* r_2),
\]

so that \( z^6 \notin (x, y)R \). This contradicts our earlier assertion that \( z^6 \notin (x, y)R \). So there can not be a Frobenius splitting of \( R \).
3. Local Frobenius Splitting is an open condition

First, let us record the easy fact that global Frobenius splitting always implies local Frobenius splitting:

**Lemma 3.1.** Let $X$ be a Frobenius split scheme.

(a) For every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ is Frobenius split.
(b) For every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is Frobenius split.
(c) The localization of a Frobenius split ring at any multiplication system is Frobenius split.

**Proof.** We prove the last statement first: if $F_*R \xrightarrow{\pi} R$ is a splitting of Frobenius for $R$, then because Frobenius commutes with localization (Lemma 1.15), we can tensor with $W^{-1}R$ to get an induced splitting $F_*(W^{-1}R) \rightarrow W^{-1}R$ of Frobenius for $W^{-1}R$. The first two statements follow by considering that if a composition $\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_X$ is the identity map, then this is true on every open set $U$, and, by taking a direct limit, at every stalk. But because Frobenius commutes with localization, we know that $(F_*\mathcal{O}_X)(U)$ is $F_*(\mathcal{O}_X(U))$ and also that the stalk $(F_*\mathcal{O}_X)_x$ is $F_*\mathcal{O}_{X,x}$. \hfill \Box

The locus of points where a scheme is locally Frobenius split is open. Moreover, for affine schemes, local and global Frobenius agree:

**Proposition 3.2.** Let $X$ be a Noetherian $F$-finite scheme of prime characteristic.

(a) If $X$ is Frobenius split at $x \in X$, then there is an affine neighborhood $U$ of the point $x$ which is Frobenius split. That is, the locus of Frobenius split points of $X$ is open.
(b) If $X$ is affine, then $X$ is Frobenius split if and only if every local ring of $X$ is Frobenius split.
(c) In the local case, an $F$-finite Noetherian local ring $(R, m)$ is Frobenius split if and only if its completion $(\widehat{R}, \widehat{m})$ is.

Before proving Proposition 3.2, we record a useful general technique for establishing splitting:

**Lemma 3.3.** Let

\[
R \xrightarrow{\iota} M \quad 1 \mapsto m,
\]

be an $R$-module homomorphism. Then $\iota$ splits if and only if the natural $R$-module map

\[
\text{Hom}_R(M, R) \xrightarrow{\Psi} R \quad \phi \mapsto \phi(m).
\]

is surjective.

---

1The proof uses only that $F_*R$ is finitely presented; we don’t need Noetherianness.
Proof of Lemma 3.3. To find a splitting of $R \xrightarrow{\iota} M$ is to find an $R$-module map $\phi \in \text{Hom}_R(M, R)$ such that $\phi(\iota(1_R)) = 1_R$. Indeed, in this case $\phi(\iota(r)) = r\phi(\iota(1_R)) = r\phi(m) = r \cdot 1_R = r$, so the composition $\phi \circ \iota$ is the identity on $R$. Clearly this is equivalent to the surjectivity of the natural $R$-module map $\text{Hom}_R(M, R) \xrightarrow{\Psi} R$ sending each $\phi$ to $\phi(m)$. □

Remark 3.4. Lemma 3.3 implies that the $R$-module $R \text{im} \Psi$ is an obstruction to the splitting of $\iota$. If $M$ is finitely presented, then $\text{Hom}_R(F_*R, R) \otimes_R R \text{im} \Psi \cong \text{Hom}_{R_p}(F_*R_p, R_p)$ for all $P \in \text{Spec } R$, and the localization of $\Psi$ at $P$ produces

$$
\text{Hom}_{R_p}(F_*R_p, R_p) \xrightarrow{\Psi_p} R_p \quad \phi \mapsto \phi(F_*1).
$$

By Lemma 3.3 $\text{Spec } R$ is split at $P$ if and only if $\Psi_P$ is surjective. So the cokernel of $R \text{im} \Psi$ can viewed as the obstruction module for Frobenius splitting: the support of the module $R \text{im} \Psi$, namely, the closed set $\mathbb{V}(\text{im } \Psi)$, is precisely the set of prime ideals for which Frobenius is not split. In particular, if $\text{Spec } R$ is split at some point $Q \in \text{Spec } R$, then there is some open set of $\text{Spec } R$, and hence of $X$, which is Frobenius split.

The second statement follows as well: the ring $R$ is Frobenius split if and only if the cokernel module $R \text{im} \Psi$ is zero. But a finitely generated module is zero if and only if it is zero at every maximal (equivalently, prime) ideal of $R$. That is $R$ is Frobenius split if and only if it is Frobenius split at every maximal (respectively, prime) ideal of $R$.

For the final statement, we need only observe that the “evaluation at 1 map” $\Psi$ is surjective if and only if it is surjective after tensoring with $\widehat{R}$— indeed, the cokernel of $\widehat{R} \text{im} \Psi$, namely, the closed set $\mathbb{V}(\text{im } \Psi)$, is precisely the set of prime ideals for which Frobenius is not split. In addition, under our hypotheses that $F_*R$ is finitely generated over a Noetherian ring,

$$
\text{Hom}_R(F_*R, R) \otimes_R \widehat{R} \cong \text{Hom}_{\widehat{R}}(F_*R \otimes \widehat{R}, \widehat{R}) \cong \text{Hom}_{\widehat{R}}(F_*\widehat{R}, \widehat{R}).
$$

So $R$ is Frobenius split if and only if $\widehat{R}$ is Frobenius split. □
It will often be useful to iterate Frobenius. For this reason, we record the next fact for future reference.

**Proposition 3.5.** Let $X$ be a scheme of prime characteristic $p$. Then the following are equivalent:

(a) $X$ is Frobenius split;

(b) There exists an $e > 0$ such that the iterated Frobenius $O_X \to F^e_*O_X \quad r \mapsto r^{p^e}$ splits in the category of $O_X$-modules.

(c) For every $e > 0$, the iterated Frobenius $O_X \to F^e_*O_X$ splits in the category of $O_X$-modules.

Likewise, the three statements are equivalent also if the word “Frobenius split” is replaced by “locally Frobenius split at a point $x$.”

**Proof.** It is clear that (a) implies (b).

Assume (b). Let $\pi : F^e_*O_X \to O_X$ be a splitting of $O_X \to F^e_*O_X$. Because $F^e$ is the composition of $F$ with $F^{e-1}$, we have that the composition

$$O_X \xrightarrow{F} F_*O_X \xrightarrow{F^{e-1}} F^e_*O_X \xrightarrow{\pi} O_X$$

is the identify map on $O_X$. This means that the composition

$$F_*O_X \xrightarrow{F^{e-1}} F^e_*O_X \xrightarrow{\pi} O_X$$

is a splitting of Frobenius on $X$. Therefore (a) and (b) are equivalent.

Clearly (c) implies (a). Assume (a). Let $\pi : F_*O_X \to O_X$ be a splitting of the Frobenius map for $X$, so that the composition

$$O_X \to F_*O_X \xrightarrow{\pi} O_X$$

is the identity map, where here the first map is the Frobenius. Applying the exact functor $F_*$, we see that

$$F_*O_X \to F^2_*O_X \xrightarrow{F^e_*\pi} F_*O_X$$

is also the identity map. Precomposing with $F$ and post-composing with $\pi$, we have that

$$O_X \to F_*O_X \to F^{2}_*O_X \xrightarrow{F^e_*\pi} F_*O_X \xrightarrow{\pi} O_X$$

is the identity map. Therefore the composition $\pi \circ F_*\pi : F^2_*O_X \to O_X$ is a splitting of the composition $O_X \to F_*O_X \to F^2_*O_X$, which is of course the composition of Frobenius with itself, $F^2$. Thus $F^2$ splits, and by induction on $e$, we see that $F^e$ splits for every $e \geq 1$. This shows that (a) implies (c), whence (a) and (c) equivalent as well.

Finally, the statement for local Frobenius splitting follows immediately by allowing $X$ to the local scheme $\text{Spec } O_{X,x}$ at a point. $\square$
4. F-regularity

F-regularity is a strong form of Frobenius splitting; the “F” stands for Frobenius, of course.

First introduced by Mel Hochster and Craig Huneke in their theory of tight closure, it has since become important in birational algebraic geometry. We define it first in the affine setting:

**Definition 4.1.** Let $R$ be an $F$-finite ring of prime characteristic. We say that $R$ is strongly F-regular if for every $c \in R$, not in any minimal prime of $R$, there exists $e \in \mathbb{N}$ such that the $R$-module map

$$R \rightarrow F^e R \quad 1 \mapsto F^e c$$

splits as a map of $R$-modules. Equivalently, for every such $c$, there exists $e \in \mathbb{N}$ and $\pi \in \text{Hom}_R(F^e R, R)$ such that $\pi(F^e c) = 1$.

Observe that the map $R \rightarrow F^e R$ in Definition 4.1 is not the Frobenius map, nor indeed, any map of rings. Rather, because $R$ is a free $R$-module of rank one, we can uniquely determine an $R$-module map from $R$ to any other module by simply specifying the image of $1_R$. Here, we are sending $1$ to $c$. This determines the image of every $r \in R$ by $R$-linearity: namely $r \mapsto r F^e c = F^e(r^p c)$.

Alternatively, an $F$-finite ring $R$ is strongly F-regular if for all $c$ (not in any minimal prime), there is an $e$ such that $F^e R$ decomposes as a direct sum $F^e R \cong RF^e c \oplus M$ for some $R$-module $M$.

Like Frobenius splitting, there are two different ways to generalize strong F-regularity to arbitrary schemes: local F-regularity and global F-regularity. A locally F-regular scheme is simply one whose local rings are all F-regular. For global F-regularity, the role of $c$ will be played by sections of invertible sheaves. We postpone the precise definition of global F-regularity until Section 6.

**Remark 4.2.** In these notes, we drop the adverb “strongly” and refer to the property in Definition 4.1 simply as F-regularity; this same abuse of terminology appears in the literature, especially in algebraic geometry papers. However, we caution the reader that Hochster and Huneke introduced three flavors of F-regularity, all conjectured to be equivalent: strong F-regularity (Definition 4.1), weak F-regularity (all ideals are tightly closed) and F-regularity (all ideals tightly closed in all localizations). Originally, they were primarily interested in the property that all ideals are tightly closed. (All three have been proved to be equivalent, for example, for Gorenstein rings [HH89] and graded rings [LS99], but this remains open in general.) Meanwhile, strong F-regularity has found many applications outside tight closure theory, and is arguably now the better known concept.

---

2We will be saying just “F-regular” for short.
Remark 4.3. Students raised the natural question: why the restriction, in the definition of F-regular, that $c$ is not in any minimal prime? Note that if $c$ is a zero-divisor, the map $R \xrightarrow{\iota} F^e_*R$ sending $1 \mapsto F^e_*c$ can never split. Indeed, if $rc = 0$ for some non-zero $c, r$, then $\iota(r) = rF^e_*c = F^e_*(r^p c) = 0$, which means that we can never find $\phi \in \text{Hom}_R(F^e_*R, R)$ such that $\phi \circ \iota$ is the identity on $R$. On the other hand, the splitting in the $c = 1$ case (that is, the splitting of Frobenius) implies that $R$ is reduced, so not being in any minimal prime is the same as not being a zero divisor.

4.1. Properties of F-regular rings. We summarize most of what we intend to prove about (strong) F-regularity in the local setting in one Meta-theorem:

Theorem 4.4. Let $R$ be an F-finite Noetherian ring (of characteristic $p > 0$).

(a) If $R$ is regular, then $R$ is F-regular.
(b) If $R$ is F-regular, then $R$ is Frobenius split.
(c) If $S \hookrightarrow R$ is an inclusion of rings which splits as a map of $S$-modules, then if $R$ is F-regular, so is $S$.
(d) If $R$ is F-regular, then $R_Q$ is F-regular for every $Q \in \text{Spec} R$.
(e) Conversely, $R$ is F-regular if $R_Q$ is F-regular for all maximal (equivalently prime) ideals $Q \in \text{Spec} R$.
(f) A local ring $(R, m)$ is F-regular if and only if the completion $\hat{R}$ of $R$ at its maximal ideal is F-regular.
(g) The locus of points $P$ in Spec $R$ where $R_P$ is F-regular is open.
(h) If $R$ is F-regular, then $R$ is normal. In particular, a local F-regular ring is a domain.
(i) If $R$ is F-regular, then $R$ is Cohen-Macaulay.
(j) If $R$ is F-regular, then for any normal scheme $X$ mapping properly and birationally to Spec $R$, $H^i(X, \mathcal{O}_X) = 0$ (that is, $R$ is pseudo-rational).

Statements (a) and (c) in Proposition 4.4 tell us that locally F-regular schemes are relatively abundant: any F-finite regular scheme is locally F-regular; for example, any smooth variety of characteristic $p$ is locally F-regular. The second result tells us that quotients of regular schemes by finite groups (whose orders are not divisible by $p$) and many other non-regular schemes are also locally F-regular.

Facts (d), (e), (f), and (g) tell us that local F-regularity is a local condition—for (f) this is a somewhat philosophical statement, referring to the fact the the completion gives a neighborhood smaller than any Zariski neighborhood. Theorem 4.13 provides an important tool for checking F-regularity that will enable us to check splitting only for one, well-chosen, $c$.

The last three facts give an impressive list of “nice” features of locally F-regular schemes—that is, strong restrictions on their singularities. These will take some effort to prove (we will first review the definitions).
4.1.1. The Proofs of (a) through (e).

**Proposition 4.5.** If \((R, \mathfrak{m})\) is a Noetherian local \(F\)-finite regular ring, then \(R\) is strongly \(F\)-regular.

**Proof.** Fix \(c \in R\) not in any minimal prime. Choose \(e \gg 0\) so that \(c \notin \mathfrak{m}^p\); this is possible since the intersection of all powers of \(\mathfrak{m}\), and hence all \(\mathfrak{m}^p\), is zero. This choice of \(e\) ensures that \(F^e c \notin F^e \mathfrak{m}^p = \mathfrak{m} \cdot F^e R\), so that the class of \(F^e c\) is non-zero in \(F^e R/\mathfrak{m} F^e R\). By Nakayama’s Lemma, we conclude that \(F^e c\) is part of a minimal generating set for \(F^e R\), and hence part of a basis for the free module \(F^e R\) over \(R\). The projection from \(F^e R\) onto the free \(R\)-summand spanned by \(F^e c\) induces an \(R\)-module homomorphism \(F^e R \to R\) sending \(F^e c\) to 1 we need. \(\Box\)

The next proposition is proved similarly to the analogous statement for Frobenius splitting (Proposition 2.4), so we leave the proof as an exercise.

**Proposition 4.6.** Suppose that \(S \hookrightarrow R\) is an inclusion of Noetherian rings that splits in the category of \(S\)-modules. Then if \(R\) is strongly \(F\)-regular, so is \(S\).

**Remark 4.7.** An analog of Proposition 4.6 is an open problem in characteristic zero for KLT singularities.

**Proposition 4.8.** Every \(F\)-regular ring is Frobenius split.

**Proof.** Taking \(c\) to be 1, we know that there exists an \(e\) such that \(R \hookrightarrow F^e R\) sending 1 \(\mapsto F^e 1\) splits. Since \(F^e\) factors through \(F\), it follows that \(R\) is Frobenius split (Proposition 3.5). \(\Box\)

**Remark 4.9.** A stronger form of Proposition 4.8 holds: Suppose that \(R\) admits one \(c\) such that the \(R\)-module map

\[ R \to F^e R \quad 1 \mapsto F^e c \]

splits for some \(e\). Then we claim that this is enough to imply Frobenius splitting for \(R\).

To see this, fix \(\phi \in \text{Hom}_R(F^e R, R)\) such that \(\phi(F^e c) = 1\). Consider the composition

\[ F^e R \xrightarrow{F^e c} F^e R \xrightarrow{\phi} R \]

where the first map is multiplication by \(F^e c\). It is easy to check this composition is an \(R\)-module map sending 1 \(\mapsto F^e c \mapsto \phi(F^e c) = 1\). This map splits the Frobenius map \(F^e\), hence \(R\) is Frobenius split (Proposition 3.5).

We record this useful property as a new definition:

**Definition 4.10.** Let \(R\) be an \(F\)-finite ring of prime characteristic, and let \(c \in R\) be not in any minimal prime. We say that \(R\) is **eventually Frobenius split along** \(c\) if there exists \(e \in \mathbb{N}\) such that the \(R\)-module map \(R \to F^e R\) sending 1 \(\mapsto F^e c\) splits.
Remark 4.9 shows that eventual Frobenius splitting along any $c$ implies Frobenius splitting.

Note also that $R$ is F-regular if and only if it is eventually Frobenius split along every non-zero-divisor $c$.

**Remark 4.11.** The argument in Remark 4.9 also shows that if $R$ is eventually Frobenius split along some $c$, then $R$ is eventually Frobenius split along any $d$ dividing $c$.

**Proposition 4.12.** Let $R$ be an F-finite Noetherian ring.

(a) If $R$ is F-regular, then so is $W^{-1}R$ for any multiplicative system $W$.

(b) Conversely, if $R_m$ is strongly F-regular for each maximal $m \in \text{Spec } R$, then $R$ is strongly F-regular.

**Proof.** The first statement is easy: given $c \in W^{-1}R$ not in any minimal prime of $W^{-1}R$, we need to show $W^{-1}R$ is eventually Frobenius split along $\frac{c}{w}$. It suffices to check this for $\frac{c}{1} \in W^{-1}R$ (Remark 4.11). Note that we may assume $c \in R$ is not in any minimal prime of $R$. Since $R$ is F-regular, we can find $e \in \mathbb{N}$ and $\phi \in \text{Hom}(F_e R, R)$ such that $\phi(F_e c) = 1$. The localization of $\phi$ at $W$ gives the desired splitting for $W^{-1}R$.

The second statement is proved similarly to the analogous statement for Frobenius splitting (Proposition 3.2): we consider the surjectivity of the “evaluation at $F_e c$ map”

$$\text{Hom}(F_e R, R) \to R \quad \phi \mapsto \phi(F_e c).$$

Since each $R_m$ is F-regular, we know each $R_m$ is Frobenius split (Proposition 4.8), and hence $R$ is Frobenius split (Proposition 3.2). Note also that if we have a map $\phi : F_e R \to R$ which sends $F_e c \mapsto 1$ in any Frobenius split ring, then we can replace $e$ by a larger $e$ by post-composing with Frobenius splittings. That is, the map

$$F_e^{e+1} R \xrightarrow{F_e \phi} F_e R \xrightarrow{\pi} R \quad F_e^{e+1} c \mapsto F_e c \mapsto 1$$

is a splitting of the corresponding map $R \mapsto F_e^{e+1} R$ sending $1 \mapsto F_e^{e+1} c$. Repeating, we have a splitting for any larger $e$.

Now take $c \in R$ not in any minimal prime of $R$. Its image in $R_m$, which we also denote by $c$, is not in any minimal prime of $R_m$. By hypothesis, for each $m \in \text{Spec } R$, the $R$-module map

$$\text{Hom}_R(F_e c R, R)_m \xrightarrow{\text{eval}_c} R_m \quad \phi \mapsto \phi(F_e c)$$

is surjective for $e \gg 0$. Thus for each $m$, there exists an $e = e_m \in \mathbb{N}$ and a neighborhood $U_m$ of $m$ such that $\text{Hom}_R(F_e c R, R)_n \xrightarrow{\text{eval}_c} R_n$ is surjective for all $n \in U_m$. These $U_m$ cover $\text{Spec } R$. But because $\text{Spec } R$ is quasi-compact, we may pick a finite sub-cover, and then some $e \geq e_m$ for each of the finitely many $m$ indexing the finite subcover. But then the $R$-module map $\text{Hom}_R(F_e R, R) \xrightarrow{\text{eval}_c} R$ is surjective at every point of $R$, and hence surjective. That is, $R$ is F-regular. \hfill \Box
4. F-REGULARITY

4.2. Test elements. One difficulty in verifying the definition of F-regularity is that one must check a splitting condition for all \( c \). The next theorem greatly increases our ability to verify F-regularity by allowing us to check just one \( c \):

**Theorem 4.13.** Let \( R \) be an F-finite Noetherian ring. Suppose that \( d \in R \) is such that \( R[d^{-1}] \) is strongly F-regular. If there exists \( e \in \mathbb{N} \) and an \( \mathcal{R} \)-module map \( \phi : F^e\mathcal{R} \to \mathcal{R} \) satisfying \( \phi(F^e d) = 1 \), then \( \mathcal{R} \) is strongly F-regular.

Put differently, Theorem 4.13 says that to check whether \( \mathcal{R} \) is F-regular, it is enough to check that \( \mathcal{R} \) is eventually Frobenius split along some \( d \in \mathcal{R} \) (Definition 4.10) such that \( \mathcal{R}[d^{-1}] \) is F-regular.

**Proof.** First note that because \( \mathcal{R} \) is Frobenius split along \( d \), then \( \mathcal{R} \) is Frobenius split (by precomposing \( \phi \) with \( F^e \); Remark 4.9).

Take \( c \in \mathcal{R} \) not in any minimal prime. We need to split the \( \mathcal{R} \)-submodule generated by \( F^f c \) off from \( F^f \mathcal{R} \) for some large \( f \). For this, it suffices to show that the “evaluation at \( c \) map”

\[
\text{Hom}_R(F^f \mathcal{R}, \mathcal{R}) \xrightarrow{\Phi_f} \mathcal{R} \quad \phi \mapsto \phi(F^f c)
\]

is surjective for some \( f \gg 0 \) (Lemma 3.3). Because \( \mathcal{R}[d^{-1}] \) is strongly F-regular, the map \( \Phi_f \) (for large enough \( f \)) becomes surjective after tensoring with \( \mathcal{R}[d^{-1}] \). This means that \( d^m \in \text{Image}(\Phi_f) \) for some \( f \gg 0 \) and some \( m > 0 \) (Remark 3.4). Without loss of generality, we may assume that \( m = p^l \) for some integer \( l \). In particular, there exists \( \psi \in \text{Hom}_R(F^f \mathcal{R}, \mathcal{R}) \) such that \( \psi(F^f c) = d^p^l \).

Now let \( \kappa : F^l \mathcal{R} \to \mathcal{R} \) be a Frobenius splitting and consider the composition

\[
F^e + \psi F^e \mathcal{R} \xrightarrow{\psi} F^e \mathcal{R} \xrightarrow{F^e \kappa} F^e \mathcal{R} \xrightarrow{\phi} \mathcal{R}
\]

The composition \( \phi \circ (F^e \kappa) \circ (F^e + l \psi) \) gives a splitting of the map \( \mathcal{R} \to F^{e+l+f} \mathcal{R} \) sending \( 1 \mapsto F^{e+l+f} c \). Thus \( \mathcal{R} \) is F-regular. \( \square \)

The element \( c \) which we are using to “test” F-regularity can be informally called a test element. This is closely related to (but not exactly the same as) a formal definition of test elements we will introduce later.

Using Theorem 4.13 we can complete the proofs of (f) and (g).

**Proof of (f).** Let \((\mathcal{R}, m)\) be an F-finite Noetherian ring. Fix any \( c \) not in any minimal prime of \( \mathcal{R} \). The (the image of) \( c \) in \( \hat{\mathcal{R}} \) is also not in any minimal prime of \( \hat{\mathcal{R}} \) (since \( \mathcal{R}/c \mathcal{R} \) and \( \hat{\mathcal{R}}/c \hat{\mathcal{R}} \approx \hat{\mathcal{R}}/c \hat{\mathcal{R}} \) have the same dimension).

Consider the \( \mathcal{R} \)-module maps

\[
\text{Hom}_R(F^e \mathcal{R}, \mathcal{R}) \to \mathcal{R} \quad \phi \mapsto \phi(F^e c)
\]
which are surjective—have trivial cokernel—if and only if they are surjective after tensoring with the completion, by the faithful flatness of \( \hat{R} \).

This produces

\[
\text{Hom}_\hat{R}(F_*^e \hat{R}, \hat{R}) \longrightarrow \hat{R} \quad \phi \mapsto \phi(F_*^e c).
\]

Clearly if \( \hat{R} \) is F-regular, then these maps are surjections for large \( e \), so that also \( R \) is F-regular, as \( c \) was an arbitrary element of \( R \).

Conversely, assume that \( R \) is F-regular. Our argument above gives splittings for all \( c \) of a certain type in \( \hat{R} \), namely those coming from \( R \). The only issue to consider is whether knowing this surjectivity (over \( \hat{R} \)) for all \( c \in R \) is enough: might there be elements \( c \) in \( \hat{R} \) but not \( R \) for which we need to verify the splitting?

This difficulty is overcome by Theorem 4.13: we only need to find an element \( c \in R \) such that \( \hat{R}[c^{-1}] \) is regular (hence F-regular).

To find such \( c \), first note that because \( R \) is Frobenius split, it is reduced. The regular locus of a reduced ring is always non-empty, since localizing at a minimal prime produces a zero-dimensional local reduced ring—that is, a field—which is of course regular.

Now since \( F_* R \) is finitely generated, the locus is \( \text{Spec} R \) where it is free is open (see [Har77][II, Ex 5.7]). Thus there is some basic open affine \( \text{Spec} R[c^{-1}] \) where \( F_* R \otimes_R R[c^{-1}] \cong F_* R[c^{-1}] \) is free for all points in \( \text{Spec} R[c^{-1}] \). By Kunz’s theorem, because this locus is non-empty, so there must be some \( c \in R \) such that \( \hat{R}[c^{-1}] \) is regular (note that \( c \) can be assumed not in any minimal prime by Lemma 4.14).

But now tensoring with \( \hat{R} \), we have

\[
\hat{R} \otimes_R R[c^{-1}] \otimes_R F_* R \cong (\hat{R} \otimes_R R[c^{-1}]) \otimes_R (\hat{R} \otimes_R F_* R) \cong \hat{R}[c^{-1}] \otimes_R F_* \hat{R} \cong F_* (\hat{R}[c^{-1}])
\]

is locally free over \( \hat{R}[c^{-1}] \). So Kunz’s theorem implies that \( \hat{R}[c^{-1}] \) is regular, as claimed. So now we can invoke Theorem 4.13 to check that \( \hat{R} \) is strongly F-regular: it is enough to find an \( e \in \mathbb{N} \) for which \( \hat{R} \longrightarrow F_* \hat{R} \) sending \( 1 \mapsto F_* c \) splits. But as we have seen, this follows from the splitting over \( R \).

\[\Box\]

**Lemma 4.14.** If \( R \) is a reduced Noetherian ring whose regular locus is open (for example, an F-finite ring), then there is \( c \in R \) not in any minimal prime such that \( R_c \) is regular.

**Proof.** Let \( P \) be a minimal prime of \( \text{Spec} R \). Then since \( R_P \) is reduced, local and dimension zero, it must be a field (hence regular). This tells us that the regular locus of \( \text{Spec} R \), which we know to be open, must have complement \( \mathbb{V}(I) \), where \( I \) has height at least one. Since \( I \not\subset P \) for any minimal prime \( P \), the prime avoidance lemma allows us to choose \( c \in I \) but not in any minimal prime. This \( c \) satisfies \( R_c \) is regular.

\[\Box\]

The proof of the openness of the F-regular locus uses Theorem 4.13 in a similar way:
Proof of (g). Let $R$ be an $F$-finite Noetherian ring. Assume that $m \in \text{Spec } R$ is such that $R_m$ is $F$-regular. It suffices to find a basic open affine set $\text{Spec } R[g^{-1}]$, for some $g \in R$, which both contains $m$ and is $F$-regular.

To this end, first note that since $R_m$ is $F$-regular, it is reduced. Since the reduced locus of $\text{Spec } R$ is open, we can replace $R$ by an affine open neighborhood of $m$ which is reduced. So without loss of generality, we can assume $R$ is reduced. Since $R$ is $F$-finite, Kunz’s theorem ensures that the regular locus is open (Corollary 4.3). In this case, we can use Lemma 4.14 to find $d \in R$, not in any minimal prime, such that $R[d^{-1}]$ is regular (and hence $F$-regular). Thus $d$ is a test element for strong $F$-regularity for $R$ in the sense of Theorem 4.13.

For any $g \in R$, localizing $R[d^{-1}]$ further at $g$, we still have an $F$-regular ring, so $d$ can be used as test element for $R[g^{-1}]$ as well.

Consider the “evaluation at $d$” map

$$\text{Hom}_R(F^e_* R, R) \xrightarrow{\Psi_e} R \phi \mapsto \phi(F^e_* d).$$

We know that for large enough $e$, this map is surjective after localizing at $m$, since $R_m$ is $F$-regular. Thus $m$ is not in the support of $R/\text{im } \Psi_e$—that is, $m \notin \text{im } \Psi_e$. So take $g \in \Psi_e \setminus m$, and note that

$$R[g^{-1}] \longrightarrow F^e_* R[g^{-1}] \quad 1 \mapsto F^e_* d$$

splits. Now Theorem 4.13 implies that $R[g^{-1}]$ is strongly $F$-regular. Since $g \notin m$, the open set $\text{Spec } R[g^{-1}]$ gives the needed open neighborhood at $m$ entirely contained in the $F$-regular locus. This completes the proof that the $F$-regular locus is open.

4.3. $F$-regular Rings are Normal. We now review normality and prove that $F$-regular rings are normal.

Definition 4.15. Let $A \hookrightarrow B$ be any inclusion of rings. We say an element $b \in B$ is integral over $A$ if $b$ satisfies an monic polynomial with coefficients in $A$:

$$b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0$$

where $a_i \in A$. The ring $A$ is said to be integrally closed in $B$ if whenever $b \in B$ is integral over $A$, then $b \in A$.

Definition 4.16. A reduced ring $R$ is normal if it is integrally closed in its total quotient ring $S^{-1}R$, where $S$ is the multiplicative set of non-zero-divisors of $R$.

Normalness is usually defined only for domains, in which case normal is the same as integrally closed in its field of fractions.\(^3\)

\(^3\)For rings with infinitely many minimal primes (necessarily non-Noetherian), there are some subtleties in defining normality; and therefore, there are competing definitions that may not agree with this one. In particular, a ring satisfying Definition 4.16 may not have the property that each $R_P$ also does as well. See [Sta16, Tag 037B](https://stacks.math.columbia.edu/tag/037B)
Remark 4.17. Many basic facts about normal rings are usually proved in Math 614, including

(a) A ring $R$ with finitely many minimal primes (e.g. a Noetherian ring) is normal if and only if $R_P$ is normal for all prime (equivalently, maximal) ideals in $R$. \cite{AM69}[Prop 5.13]

(b) A normal local ring is a domain. \cite{Mat89}[p 64]

(c) A one dimensional normal Noetherian ring is regular; in particular, it is a discrete valuation ring. \cite{AM69}[Prop 9.2]

(d) A Noetherian domain is normal if and only if

$$R = \bigcap_{P \in \text{Spec } R \text{ ht}(P) = 1} R_P.$$  

\cite{Mat89}[Thm 11.5]

The third condition above has an important geometric consequence: on a normal variety $X$, the non-regular locus must have codimension at least two.

Theorem 4.18. \cite{HH90} Strongly F-regular rings are normal.

Proof of Theorem 4.18. Assume that $R$ is F-regular. Then $R$ is Frobenius split, and hence reduced.

Fix an element $x/y$ in the total quotient ring of $R$ integral over $R$. We must show that $y$ divides $x$ in $R$. Since $x/y$ is integral over $R$, there is an integral equation

$$(x/y)^m + r_1(x/y)^{m-1} + \cdots + r_m = 0,$$

where each $r_j$ is in $R$. Consider the ring

$$T = R[x/y] \cong R[X]/\langle X^m + r_1X^{m-1} + \cdots + r_m \rangle,$$

clearly a finite integral extension of $R$, with $R$-module generators $\{1, \frac{x}{y}, (\frac{x}{y})^2, \ldots, (\frac{x}{y})^{m-1}\}$.

Hence there is $c \in R$ not in any minimal ring such that $cT \subset R$ (for example, we can take $c = y^{m-1}$).

Since $(\frac{x}{y})^{pe} \in T$ for all $e$, we have $c(\frac{x}{y})^{pe} \in R$ for all $e \in \mathbb{N}$. That is, $cx^{pe} \in \langle y^{pe} \rangle$ in $R$ for all $e \geq 1$. Therefore we have, for all $e$, an equation of the form

$$cx^{pe} = r_e y^{pe} \tag{4.18.1}$$

where $r_e \in R$ depends on $e$ but $c$ does not. Since $R$ is F-regular, we can find $\phi \in \text{Hom}_R(F^e_* R, R)$ such that $\phi(F^e_* c) = 1$. We can then view equation (4.18.1) in the ring $F^e_* R$, so that

$$F^e_* (cx^{pe}) = F^e_* (r_e y^{pe})$$

which simplifies to

$$xF^e_* c = yF^e_* r_e.$$

Applying the $R$-linear map $\phi$, we get statements in $R$:

$$x\phi(F^e_* c) = y\phi(F^e_* r_e)$$

$$x \cdot 1 = y\phi(F^e_* r_e)$$

That is, $x \in \langle y \rangle$ in $R$. This shows that $x/y \in R$ and finishes the proof. \qed
Remark 4.19. There are various weakenings of the normality condition such as “weakly normal” and “semi-normal” which I won’t define here. The idea is that $R$ is weakly (or semi) normal if all integral elements in its total quotient ring of certain types are in $R$. It turns out that a Frobenius split ring is always weakly normal (and hence semi-normal). I will not prove this here, but if you are interested in these variants of normality, this could be the subject of a student talk.

4.4. F-regular Rings are Cohen-Macaulay. The property of Cohen-Macaulayness is so central to commutative algebra that the field has been jokingly called the “study of Cohen-Macaulayness.” The Cohen-Macaulay property is also important in algebraic geometry, representation theory and combinatorics, with many different characterizations. We briefly review some of these. See [BH93] for a more in depth discussion.

First, Cohen-Macaulay is a property of Noetherian schemes that is defined locally—meaning that a scheme $X$ is defined to be Cohen-Macaulay if all its local rings at closed points are Cohen-Macaulay. So we focus only on what it means for a local Noetherian ring $(R, m)$ to be Cohen-Macaulay.

First recall the definition of a regular sequences:

**Definition 4.20.** [BH93, Definition 1.1.1] A sequence of elements $x_1, \ldots, x_d$ in a ring $R$ is a regular sequence if $x_1$ is not a zero divisor on $R$, and the image of $x_i$ in $R/(x_1, \ldots, x_{i-1})$ is not a zero divisor on $R/(x_1, \ldots, x_{i-1})$ for each $i = 2, 3, \ldots, d$.

Regular sequences are useful for creating induction arguments using long exact sequences induced from the short exact sequences

$$0 \to R/(x_1, \ldots, x_{i-1}) \xrightarrow{x_i} R/(x_1, \ldots, x_{i-1}) \to R/(x_1, \ldots, x_i) \to 0.$$  

In algebraic geometry, say when $R$ is the homogeneous coordinate ring of a projective variety, this is the technique of “cutting down by hypersurface sections.” This works best when the resulting intersections contain no embedded points—which is to say, the defining equations of the hypersurfaces form a regular sequence.

We can now recall the standard textbook definition of Cohen-Macaulayness. Don’t worry if some of these characterizations are unfamiliar; for example, we will review local cohomology later when we need it.

**Definition 4.21.** [BH93, Definition 2.1.1] A local Noetherian ring $(R, m)$ is Cohen-Macaulay if any of the following equivalent conditions holds

(a) There is a regular sequence contained in $m$ of length equal to the dimension of $R$.

(b) Some system of parameters for $R$ is a regular sequence.

(c) Every system of parameters for $R$ is a regular sequence.

(d) The Koszul complex on some (equivalently, every) system of parameters for $R$ is exact.

(e) The local cohomology modules $H^i_m(R)$ are all zero for $i < \dim R$. 


Remark 4.22. If \((R, m)\) is a local ring, by tensoring the sequences (4.20.1) by \(\hat{R}\), we see that a system of parameters \(x_1, \ldots, x_d\) for \(R\) forms a regular sequence in \(R\) if and only if their images form a regular sequence on \(\hat{R}\). In particular, \(R\) is Cohen-Macaulay if and only if \(\hat{R}\) is Cohen-Macaulay.

4.4.1. An Alternative Characterization Using Noether Normalization. Because the textbook definition can seem somewhat mysterious, we discuss an alternative characterization. We use a fact proved by Kannanpan in class:

Proposition 4.23. Let \(R\) be a local Noetherian ring. Suppose that \(R\) admits a regular local subring \(A\) such that \(A \hookrightarrow R\) is a finite local homomorphism. Then \(R\) is Cohen-Macaulay if and only if \(R\) is free as module over \(A\).

Proof. For details, see Kannanpan’s notes. The point is our hypothesis ensures that a minimal generating set for the maximal ideal of \(A\) will be a system of parameters for \(R\), so \(R\) is Cohen-Macaulay if and only if it has depth \(d\) (=dimension \(R = \) dimension \(A\)) as a module over \(A\). We can then apply the Auslander-Buchsbaum theorem to the \(A\)-module \(R\) to conclude that this happens if and only if \(R\) has projective dimension zero over \(A\)—that is, if and only if \(R\) is free over \(A\). \(\square\)

Now to understand Cohen-Macaulayness of a local ring \(R\), it suffices to consider the complete local case (Remark 4.22). When \((R, m, K)\) is a complete local ring over a field, an analog of Noether Normalization holds: there is a finite integral extension \(A = K[[x_1, x_2, \ldots, x_d]] \hookrightarrow R\) from some power series sub-algebra. Indeed, any system of parameters \(\{x_1, \ldots, x_d\}\) for \(R\) will generate a power series subalgebra \(A\) such that \(R \hookrightarrow\) is a finite integral extension. This follows from the Cohen Structure theorem: we see in this case that \(R/(x_1, \ldots, x_d)\) is finite dimensional over \(K\), and we leave as an exercise that a \(K\)-basis lifts to a generating set for \(R\) over \(A\).

Proposition 4.23 then becomes:

Proposition 4.24. Let \(R\) be a complete local ring, and represent \(R\) as a finite extension of a power series subring \(A = K[[x_1, x_2, \ldots, x_d]]\). Then \(R\) is Cohen-Macaulay if and only if \(R\) is free as an \(A\)-module.

This gives a convenient way to represent elements of a complete Cohen-Macaulay ring \(R\): once we fix some free basis for \(R\) over \(A\), every element of \(R\) can be uniquely expressed as a linear combination of these with coefficients in the power series ring \(A\).

---

4By definition, a homomorphism of local rings \((A, m_A) \xrightarrow{\phi} (B, m_B)\) is local if \(\phi(m_A) \subset M_B\).

5The Auslander-Buchsbaum theorem says that if \(M\) is an module over a Noetherian local ring \(R\), and \(M\) has finite projective dimension, then that projective dimension is equal to the length of the longest regular sequence on \(R\) minus the length of the longest regular sequence on \(M\).
Remark 4.25. Alternatively, we can work in the graded-local setting, and take \((R, m)\) to be an \(\mathbb{N}\)-graded ring \(R\), finitely generated over its zero-th graded piece \(R_0\) (a field). In this case, \(m\) denotes the unique homogenous maximal (or irrelevant) ideal of \(R\).

Every such graded ring \(R\) admits a Noether Normalization: that is, some (graded) polynomial subalgebra \(A = k[x_1, x_2, \ldots, x_d] \hookrightarrow R\) over which \(R\) is a finite integral extension. Again, the \(x_i\)'s can be taken to be any homogeneous system of parameters.

Again, the Cohen-Macaulayness of \(R\) is equivalent to the freeness of \(R\) as a module over \(A = k[x_1, x_2, \ldots, x_d]\) (Proposition 4.23).

Remark 4.26. Two easy fact about Cohen-Macaulay rings, both of which follow easily from the definitions:

(a) A regular ring is Cohen-Macaulay;
(b) If \(f \in m\) is a non-zero divisor on a local ring \((R, m)\), then \(R\) is Cohen-Macaulay if and only if \(R/\langle f \rangle\) is.

The proof of (a) is easy: reduce to the local case, and then show that a minimal generating set for the maximal ideal is a regular sequence by observing that each successive \(R/\langle x_1, \ldots, x_i \rangle\) is a domain. Likewise, (b) is easy because \(f\) can be taken to be part of a system of parameters for \(R\).

Theorem 4.27. [HH90] Every F-regular ring is Cohen-Macaulay.

We will need the following lemma:

Lemma 4.28. If \(B \hookrightarrow R\) is a finite integral extension of domains, then there exists \(\phi \in \text{Hom}_B(R, B)\) such that \(\phi(1_R) \neq 0\).

Proof. Let \(L\) be the fraction field of \(B\). Note that \(L \otimes_B R\) is a finite dimensional \(L\) vector space. So \(L \otimes_B \text{Hom}_B(R, B) \cong \text{Hom}_L(L \otimes_B R, L)\) clearly admits a projection \(L \otimes_B R \rightarrow L\) sending \(1_L \otimes 1_R\) to \(1\). Restricting to \(R \hookrightarrow L \otimes_B R\), we have a \(B\)-linear map \(R \rightarrow L\) sending \(1_R \mapsto 1_L\). Since \(R\) is finitely generated over \(B\), say by \(r_1, \ldots, r_t\), we can multiply by some non-zero \(b \in B\) to “clear denominators” of each of the finitely many \(\psi(r_i) \in L\). The map \(\phi = b\psi \in \text{Hom}_B(R, B)\) has the desired property. \(\square\)

Proof of Theorem 4.27. Without loss of generality, we can assume that \((R, m, K)\) is a complete local F-regular ring. Since F-regular rings are normal, and normal local rings are domains, we can assume also that \(R\) is a domain.
Let $x_1, \ldots, x_d$ be a system of parameters for $R$. If these are not a regular sequence, then we have some relation $zx_i \in (x_1, \ldots, x_{i-1})$, where $z \notin (x_1, \ldots, x_{i-1})$. And hence $z^{p^e} x_i^{p^e} \in (x_1^{p^e}, \ldots, x_{i-1}^{p^e})$ for all $e \geq 1$. This gives us an equation in $R$

\[(4.28.1) \quad z^{p^e} x_i^{p^e} = r_1 x_1^{p^e} + \cdots + r_{i-1} x_{i-1}^{p^e}\]

for some $r_1, \ldots, r_{i-1} \in R$ (depending on $e$).

Let $A$ be the subring of $R$ of formal power series in $x_1, \ldots, x_d$ over $K$. Thus $A$ is a complete regular subring of $R$ over which $R$ is finite. Let $B$ be the intermediate ring generated by $z$ over $A$. We have a sequence of finite integral extensions

\[A = K[x_1, x_2, \ldots, x_d] \hookrightarrow B = A[z] \hookrightarrow R.\]

All these rings are local with system of parameters $\{x_1, x_2, \ldots, x_d\}$. Note that $B$ is Cohen-Macaulay by Remark 4.26: indeed, $B \cong A[t]/(f(t)) \cong K[x_1, x_2, \ldots, x_d, t]/(f)$ where $f$ is the minimal polynomial for $z$ over $A$, so that $B$ is the quotient of a power series ring (which is Cohen-Macaulay because it is regular) by a non-zero-divisor.

Now, we can find $\phi \in \text{Hom}_B(R, B)$ such that $\phi(1_R) = c \neq 0$ (Lemma 4.28). Applying $\phi$ to equation (4.28.1), we get an equation in $B$:

\[(4.28.2) \quad cz^{p^e} x_i^{p^e} = \phi(r_1)x_1^{p^e} + \cdots + \phi(r_{i-1})x_{i-1}^{p^e}.\]

Because $\{x_1^{p^e}, x_2^{p^e}, \ldots, x_d^{p^e}\}$ is a system of parameters for the Cohen-Macaulay ring $B$, we know that $x_i^{p^e}$ is a non-zero-divisor modulo $\langle x_1^{p^e}, \ldots, x_{i-1}^{p^e} \rangle$, so that

\[(4.28.3) \quad cz^{p^e} \in \langle x_1^{p^e}, \ldots, x_{i-1}^{p^e} \rangle B \subset \langle x_1^{p^e}, \ldots, x_{i-1}^{p^e} \rangle R\]

for all $e \in \mathbb{N}$. That is, we have equations in $R$

\[(4.28.4) \quad cz^{p^e} = r_1 x_1^{p^e} + \cdots + r_{i-1} x_{i-1}^{p^e}\]

where the $r_j$ depend on $e$, but $c$ does not.

Now we can finally use the hypothesis that $R$ is $F$-regular. For large enough $e \in \mathbb{N}$, there is $\phi \in \text{Hom}_R(F_*^e R, R)$ such that $\phi(F_*^e c) = 1$. We can view Equation (4.28.5) in $F_*^e R$, where it becomes

\[(4.28.5) \quad F_*^e (cz^{p^e}) = F_*^e(r_1 x_1^{p^e} + \cdots + r_{i-1} x_{i-1}^{p^e})z F_*^e c = x_1 F_*^e r_1 + \cdots + x_{i-1} F_*^e r_{i-1}.\]

Applying the $R$-linear map $\phi$, we see that $z \in (x_1, \ldots, x_{i-1})R$, finishing the proof of the Cohen-Macaulayness of $R$. \qed

To summarize, we now have proved the following implications among classes of singularities:

\[
\{\text{Regular}\} \implies \{\text{F-regular}\} \implies \{\text{Frobenius split, Cohen-Macaulay, Normal}\}.
\]

In addition, we have shown that both Frobenius splitting and F-regularity descend to direct summands. This is all that is needed to prove the famous Hochster-Roberts theorem, at least in characteristic $p$. 


Hochster and Roberts also established this statement in characteristic zero—meaning that the regular ring contains a field of characteristic zero—by reduction to characteristic $p$, a technique we will discuss later. The corresponding statement in mixed characteristic—meaning for rings that do not contain any field—had been open for decades, until recently proved by Heitmann and Ma using Scholze’s perfectoid spaces [HM18]. We will not discuss the mixed characteristic case in this course, but an overview of the method could be a good topic for a student talk.

5. Global Frobenius Splitting and Vanishing Theorems

We now investigate the global implications of Frobenius splitting. We have seen, for example, that a smooth projective variety of characteristic $p$ is always locally Frobenius split. But global Frobenius splitting places strong restrictions on the geometry of a projective variety! This is a result of the strong vanishing theorems implied by Frobenius splitting, such as the following prototypical example:

**Theorem 5.1.** Let $X$ be a smooth projective Frobenius split variety over a field of prime characteristic $p$. For any ample invertible sheaf $\mathcal{L}$ on $X$, we have $H^n(X, \mathcal{L}) = 0$ for all $n \geq 1$.

An immediate corollary restrains the geometry of a Frobenius split curve:

**Corollary 5.2.** A smooth projective curve of genus two or more can not be Frobenius split.

**Proof of Corollary 5.2.** Let $X$ be a smooth projective curve of genus $g$ over a field of characteristic $p$. By definition, the genus of the curve is the dimension of the space $H^0(X, \omega_X)$ of global regular differential forms. Recall that an invertible sheaf on $X$ is ample if and only if its degree is positive. Since the degree of $\omega_X$ is $2g - 2$, we see $\omega_X$ is ample if and only if the genus of $X$ is at least two.

By Serre duality, $H^1(X, \omega_X)$ is dual to $H^0(X, \mathcal{O}_X)$, which is never zero since it includes, for example, all the constant functions on $X$. We conclude, by appealing to Theorem 5.1, that $X$ is never Frobenius split if its genus is larger than one. □

By contrast, curves of genus zero are always Frobenius split, as we prove in §7.1.3. In genus one—the elliptic curve case—whether or not the curve is Frobenius split is a more subtle phenomenon: an elliptic curve is Frobenius split if and only if the natural map on cohomology $H^1(X, \mathcal{O}_X) \xrightarrow{\phi} H^1(X, \mathcal{O}_X)$ induced by Frobenius is injective—that is, if and only if $X$ is ordinary (not super-singular).
Example 5.3. Corollary 5.2 generalizes to higher dimension. Let $X$ be a smooth projective variety of dimension $n$, and let $\omega_X$ be the canonical sheaf of regular $n$-forms on $X$. By Serre duality, $H^n(X, \omega_X)$ is dual to $H^0(X, \omega_X \otimes \omega_X^{-1}) = H^0(X, \mathcal{O}_X)$, which is never zero. So again invoking Theorem 5.1 we see that $X$ is never (globally) Frobenius split if $\omega_X$ is ample.

For an explicit example, consider a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d$ in $\mathbb{P}^n$. Recall the adjunction formula

$$K_X = (K_{\mathbb{P}^n} + X)|_X = -(n+1)H + dH|_X = (d-n-1)H|_X$$

describing the canonical divisor class $K_X$ for $X$. This tells us that $\omega_X$ is the restriction (pull-back) of $\mathcal{O}_{\mathbb{P}^n}(d-n-1)$ to $X$. In particular, $\omega_X$ is ample if and only if $d > n + 1$. By Theorem 5.1 such a hypersurface is never Frobenius split. Similar to the curve case, we will later see that a hypersurface of degree $d < n + 1$ is almost always Frobenius split, but that the case where $d = n + 1$, is subtle. For example, a degree three curve in $\mathbb{P}^2$ is an elliptic curve, which as we have already mentioned is Frobenius split if and only if it is ordinary.

5.1. Vanishing Theorems. Before proving Theorem 5.1 and other vanishing theorems, let us review why vanishing theorems are so important.

Ample invertible sheaves are important in algebraic geometry because they govern all embeddings of varieties into a projective space. More general invertible sheaves govern mappings to projective space which may not be embeddings, nor even defined at all points.

You may recall that if $s_0, s_1, \ldots, s_n$ are global sections of an invertible sheaf $\mathcal{L}$ on a variety $X$, then there is a rational map $X \dasharrow \mathbb{P}^n$ given by “evaluating the sections”, that is, sending $x \mapsto [s_0(x) : s_1(x) : \cdots : s_n(x)] \in \mathbb{P}^n$, suitably interpreted. In this case, the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ (over the locus where the map is defined) is $\mathcal{L}$. This map is a closed embedding if $\mathcal{L}$ is very ample and the sections are a basis for the vector space of all global sections of $\mathcal{L}$.

Every rational map from $X$ to projective space $\mathbb{P}^n$ can be described in this way for some choice of $\mathcal{L}$ and global sections $s_0, s_1, \ldots, s_n$. Thus understanding sections of invertible sheaves is a fundamental problem in algebraic geometry. In particular, we would like to know the dimension of spaces $H^0(X, \mathcal{L})$ of global sections of $\mathcal{L}$.

Some important classes of invertible sheaves for which we would like to understand the space of global sections include the ample, big, and semi-ample invertible sheaves. An invertible sheaf $\mathcal{L}$ is **ample** if and only if some power $\mathcal{L}^n$ defines an embedding into projective space (ie, is very ample), **big** if and only if some power $\mathcal{L}^n$ defines a birational map onto its image, and **semi-ample** if and only if some power $\mathcal{L}^n$ defines a regular map (ie, is defined at every point).

The Riemann-Roch Theorem is a famous formula for computing these dimensions, but because it involves the higher cohomologies $H^i(X, \mathcal{L})$ as well, it is most useful...
when we know that these higher cohomologies of $\mathcal{L}$ are all zero. This is one reason why vanishing theorems are so important in birational algebraic geometry.

To prove the vanishing of higher cohomology for ample invertible sheaves on Frobenius split projective varieties (Theorem 5.1), we will prove the following stronger theorem:

**Theorem 5.4.** Let $X$ be a Noetherian Frobenius split scheme over a field of prime characteristic $p$. Suppose that $\mathcal{L}$ is an invertible sheaf satisfying $H^n(X, \mathcal{L}^t) = 0$ for some $n \geq 0$ and all $t \gg 0$. Then also $H^n(X, \mathcal{L}) = 0$.

Observe that Theorem 5.1 follows immediately, since an ample invertible sheaf on a projective scheme always satisfies $H^n(X, \mathcal{L}^t) = 0$ for sufficiently large $t$ and all $n > 0$ (by Serre Vanishing [Har77, Prop III 5.3]).

**Proof of Theorem 5.4.** Assume that $X$ is Frobenius split. Then Frobenius, $F$, and all its iterates $F^e$, splits. Let $\pi$ be a splitting of $F^e$, so that the composition $O_X \xrightarrow{F^e} F_*O_X \xrightarrow{\pi} O_X$ is the identity map. Tensoring with the invertible sheaf $\mathcal{L}$, the composition is still the identity map:

$$L \xrightarrow{F^e} F_*O_X \otimes L \xrightarrow{\pi} L.$$  \hspace{1cm} (5.4.1)

We now analyze the coherent sheaf $F_*O_X \otimes L$. To ease the notational burden, let us write $F$ for $F^e$, remembering that $F^e$ means the map raising elements to the $p^e$-th power.

By the projection formula, $F_*O_X \otimes L$ is $F_*(F^e(L))$ [Har77, II, Ex 5.1(d)]. But it is easy to see that pulling back the invertible sheaf $L$ via (any iterate of) Frobenius produces $L^{p^e}$: indeed, the transition functions pull back under $F$ to their $p^e$-th powers, which are the transition functions for $L^{p^e}$. So the composition

$$L \xrightarrow{F} F_*L^{p^e} \xrightarrow{\pi} L,$$  \hspace{1cm} (5.4.2)

is again the identity map of sheaves. Taking cohomology, we have the identity map of abelian groups as well:

$$H^n(X, \mathcal{L}) \hookrightarrow H^n(X, F_*L^{p^e}) \twoheadrightarrow H^n(X, \mathcal{L}).$$

Since $F$ is affine, $H^n(X, F_*L^{p^e}) = H^n(X, \mathcal{L}^{p^e})$ [Har77, III Ex 4.1 or Ex 8.1]. So we see that $H^n(X, \mathcal{L})$ is a direct summand of $H^n(X, \mathcal{L}^{p^e})$, and so it will vanish if $H^n(X, \mathcal{L}^{p^e})$ does. $\square$

A version of the Kodaira vanishing theorem holds on a Frobenius split variety.

**Corollary 5.5.** Let $X$ be a smooth projective scheme over an F-finite field of prime characteristic $p$. Then for any ample invertible sheaf $\mathcal{L}$, we have $H^n(X, \omega_X \otimes \mathcal{L}) = 0$ for all $n \geq 1$.

6Actually, Cohen-Macaulay is enough: we only need Serre duality to hold.
Proof. The corollary follows by Serre duality and Serre vanishing. To show that $H^n(X, \omega_X \otimes \mathcal{L}) = 0$, we need to show that $H^{\dim X - n}(X, \mathcal{L}^{-1}) = 0$ (Serre Duality [Har77][III Thm 7.6]). By Theorem 5.4, it suffices to show that $H^{\dim X - n}(X, \mathcal{L}^{-t}) = 0$ for large $t$. Dualizing again, this is equivalent to the vanishing of $H^n(X, \omega_X \otimes \mathcal{L}^t)$ for large $t$, which follows from the ampleness of $\mathcal{L}$ (Serre Vanishing [Har77] Prop III 5.3]).

The global consequences of splitting Frobenius, and indeed the term Frobenius split, were first treated systematically by Mehta and Ramanathan in [MR85]. While inspired by Hochster and Roberts’ paper [HR74] ten years prior, which focused on the local case, Mehta and Ramanathan were motivated by the possibility of understanding the global geometry of Schubert varieties and related objects in algebro-geometric representation theory. This idea was very fruitful, leading the Indian school of algebro-geometric representation theory to many important results now chronicled in the book [BK05]. In Section 7, we formally show how the local and global points of view converge by translating global splittings of a projective variety $X$ into local splittings “at the vertex of the cone” over $X$.

6. Global F-regularity

Globally F-regular varieties have even stronger vanishing theorems, and hence stronger restrictions on their geometry. Many familiar classes of varieties are globally F-regular including toric varieties, Grassmannians and other homogeneous spaces, certain moduli spaces, and many cluster varieties. Among curves of characteristic $p$, only the genus zero curves—those isomorphic to $\mathbb{P}^1$—are globally F-regular. Among surfaces obtained by blowing up $\mathbb{P}^2$ at $d$ general points, we will prove that blowing up eight or fewer general points produces a globally F-regular variety, but blowing up nine or more points always produces a non-globally F-regular variety.

To define global F-regularity for a projective variety, we can not simply say “for all $c \in \mathcal{O}_X$” as we did for rings, because $c$ would need to be in every $\mathcal{O}_X(U)$ for open $U \subset X$. We might try to take a global section $c \in \mathcal{O}_X(X)$, since such $c$ would restrict to a section on every open set $U \subset X$, but this doesn’t work well either: the structure sheaf does not have non-constant global sections. Thus we will need to look toward non-trivial invertible sheaves on $X$ in order to find the analog of $c$.

6.1. Divisors and Invertible Sheaves. For simplicity, let $X$ be a normal Noetherian irreducible scheme (we have in mind a smooth projective variety). Let $\mathcal{K}$ denote the rational function field of $X$, and $\mathcal{K}^\times$ denote the non-zero elements (units) in $\mathcal{K}$. We also use the notation $\mathcal{K}$ and $\mathcal{K}^\times$ to denote the corresponding constant sheaves of rational functions on $X$.

\footnote{meaning, smooth connected projective curves}
Definition 6.1. A prime divisor on $X$ is a reduced irreducible closed subscheme of codimension one in $X$. A (Weil) divisor on $X$ is a formal $\mathbb{Z}$-linear combination
\[ \sum_{i=1}^{t} n_i D_i \]

of prime divisors. We say that $D$ is effective if all $n_i \geq 0$.

Because $X$ is normal, the local ring $\mathcal{O}_{X,D}$ at any prime divisor is a normal local ring of dimension one—that is, a discrete valuation ring (Proposition 4.17 c). The associated valuation $\nu_D$ on $\mathbb{K}$, which we call the “order of vanishing along $D$”, can be defined as follows: if $f \in \mathbb{K}^\times$ is regular in a neighborhood of $D$, then $f \in \mathcal{O}_{X,D}$, and we define $\nu_D(f)$ to be the largest $t$ such that $f \in m^t$, where $m$ is the maximal ideal of $\mathcal{O}_{X,D}$. If $f$ is not regular in a neighborhood of $D$, then $\frac{1}{f}$ is, and we define $\nu_D\left(\frac{1}{f}\right)$.

[When $\nu_D(f) > 0$, we say that “$f$ has a zero of order $\nu_D(f)$” along $D$. When $\nu_D(f) < 0$, we say that “$f$ has a pole of order $|\nu_D(f)|$” along $D$. And $\nu_D(f) = 0$ if and only if $f$ is a unit along $D$.]

Any non-zero rational function $f \in \mathbb{K}^\times$ has at most finitely many $D$ along which it has a zero or a pole. Thus the following definition makes sense:

Definition 6.2. For any normal Noetherian scheme $X$, the divisor associated to $f \in \mathbb{K}^\times$ is
\[ \text{div}(f) := \sum_{D \text{ prime divisor}} \nu_D(f) D. \]
A divisor of this form is said to be a principal divisor.

For any open set $U \subset X$, we can restrict a divisor $D = \sum_{i=1}^{t} n_i D_i$ to $U$ by
\[ D|_U := \sum_{i=1}^{t} n_i (D_i \cap U) \]
where we simply omit the term $n_i(D_i \cap U)$ if $D_i \cap U$ is empty.

Definition 6.3. Let $X$ be a normal Noetherian irreducible scheme, and let $D$ be a divisor on $X$. The sheaf associated to the divisor $D$ is the subsheaf of $\mathbb{K}$ defined by
\[ \mathcal{O}_X(D)(U) =: \{ f \in \mathbb{K}^\times \mid \text{div} f|_U + D|_U \geq 0 \} \cup \{0\}, \]
where by $\geq 0$ we mean that the divisor $\text{div} f|_U + D|_U$ is effective on $U$ — the coefficient of each of its prime divisors is non-negative.

Definition 6.4. Two divisors $D_1$ and $D_2$ on a normal scheme $X$ are linearly equivalent if and only if there exists $f \in \mathbb{K}^\times$ such that $D_1 - D_2 = \text{div} f$. Equivalently, there is an isomorphism $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ (given by multiplication by $f$).
Remark 6.5. Each global section \( s \in H^0(X, \mathcal{O}_X(D)) \) gives rise to a unique effective divisor linearly equivalent to \( D \), namely the divisor

\[
\text{div } s + D,
\]

which is effective by the definition of \( \mathcal{O}_X(D) \). Conversely, if \( D' \) is an effective divisor linearly equivalent to \( D \), then there is a rational function \( s \) such that \( D' = D + \text{div } s \); by definition, \( s \) is a global section of \( \mathcal{O}_X(D) \). Note that the section \( s \) uniquely determines the divisor \( D' \) but the divisor \( D' \) determines \( s \) only up to unit multiple from \( H^0(X, \mathcal{O}_X) \).

Example 6.6. If \( D \) is effective, then \( 1 \in \mathbb{K} \) is a global section of \( \mathcal{O}_X(D) \). The divisor determined by this section is \( \text{div } 1 + D = D \). So an effective divisor \( D \) determines a canonical choice of global section \( 1 \) for \( \mathcal{O}_X(D) \), which in turn recovers \( D \).

Example 6.7. Consider \( \mathbb{P}^2 \) with homogeneous coordinates \( x, y, z \). Let \( H \) be the hyperplane in \( \mathbb{P}^2 \) defined by the homogeneous coordinate \( x = 0 \). Then the global sections of \( \mathcal{O}_{\mathbb{P}^2}(H) \) are spanned by \( \{1, \frac{y}{x}, \frac{z}{x}\} \). An arbitrary global section is \( \frac{ax + by + cz}{x} \), whose associated divisor is \( \text{div}(\frac{ax + by + cz}{x}) + H \); this is the hyperplane defined by the vanishing of the linear form \( ax + by + cz \).

If \( H' \) is some other hyperplane, say defined by the homogeneous linear form \( h \), then the global sections of \( \mathcal{O}_X(H') \) are spanned by \( \{\frac{x}{h}, \frac{y}{h}, \frac{z}{h}\} \). Note that \( \frac{h}{x} \) is a rational function on \( \mathbb{P}^2 \), and that \( \text{div}(\frac{h}{x}) = H' - H \). Thus

\[
H \text{ and } H' \text{ are linearly equivalent} \quad \text{and} \quad \mathcal{O}_{\mathbb{P}^2}(H') \cong \mathcal{O}_{\mathbb{P}^2}(H)
\]

where the isomorphism given by multiplication by \( \frac{h}{x} \).

Example 6.8. Let \( D \) be an effective divisor on a normal irreducible scheme \( X \). Then the sheaf \( \mathcal{O}_X(-D) \) is the subsheaf of \( \mathcal{O}_X \) of regular functions vanishing along \( D \) (to orders prescribed by the coefficients of each prime divisor in \( D \)). Thus we can think of an effective divisor \( D \) as a subsheaf of \( X \) whose ideal sheaf is \( \mathcal{O}_X(-D) \). If \( X \) is projective, then this sheaf has no global sections, hence there are no effective divisors linearly equivalent to \( -D \).

6.1.1. Cartier Divisors and Invertible Sheaves.

Definition 6.9. A divisor \( D \) on a normal Noetherian scheme is Cartier or locally principal if there exists a cover \( \{U_\lambda\} \) of \( X \) such that each \( D_{|U_\lambda} \) is principal on \( U_\lambda \).

Proposition 6.10. On a smooth variety, every Weil divisor is Cartier.

Proof. We’ll show every Weil divisor is Cartier on any regular scheme, or more generally on any factorial scheme—that is, one with the property that all its local rings are UFDs.

First consider a prime divisor \( D \subset X \). As an irreducible closed subscheme of codimension one, its defining ideal \( \mathcal{I}_D = \mathcal{O}_X(-D) \subset \mathcal{O}_X \) is a sheaf of height one prime
Proposition 6.13. Let \( m \) be reducible, irreducible or normal) schemes. Namely we can be any divisor on \( X \). In this case, we say that \( D \) is Cartier. If \( f \) is not Cartier, since it cannot be defined by one equation at the singular point \( m = \langle x, y, z \rangle \). However, \( 2D = \text{div} x \), so \( D \) is \( \mathbb{Q} \)-Cartier.

Remark 6.12. It can happen that \( nD \) is Cartier for some divisor \( D \) which is not Cartier. For example, the prime divisor \( D = \mathbb{V}(x, z) \subset \text{Spec} k[x, y, z]/\langle xy - z^2 \rangle \) is not Cartier, since it cannot be defined by one equation at the singular point \( m = \langle x, y, z \rangle \). But \( 2D = \text{div} x \), so \( D \) is \( \mathbb{Q} \)-Cartier.

Proposition 6.13. Let \( X \) be a Noetherian normal irreducible scheme, and let \( D \) be any divisor on \( X \). The sheaf \( \mathcal{O}_X(D) \) is invertible if and only if \( D \) is Cartier. Moreover, every invertible sheaf of \( \mathcal{O}_X \)-modules is isomorphic to a sheaf of the form \( \mathcal{O}_X(D) \) for some Cartier divisor \( D \).

Proof. Suppose that \( D \) is Cartier. If \( f \in \mathcal{K}^\times \) is a local defining equation for \( D \) on some open set \( U \) (that is, \( D|_U = \text{div}_U(f) \)), then it is easy to check that \( f^{-1} \) is a generator for \( \mathcal{O}_X(D)|_U \) as an \( \mathcal{O}_U \)-submodule of \( \mathcal{K} \). This shows that \( \mathcal{O}_X(D) \) is locally free of rank one, or invertible.

Conversely, let \( \mathcal{L} \) be an invertible \( \mathcal{O}_X \)-submodule of \( \mathcal{K} \). Choose an affine open cover \( \{U_\lambda\} \) trivializing \( \mathcal{L} \), so that each \( \mathcal{L}|_{U_\lambda} \) is free of rank one over \( \mathcal{O}_{X|U_\lambda} \). Say that \( g_\lambda \in \mathcal{K} \)

\[ g_\lambda \] is a generator for \( \mathcal{O}_X(D)|_{U_\lambda} \) as an \( \mathcal{O}_{X|U_\lambda} \)-submodule of \( \mathcal{K} \). This shows that \( \mathcal{O}_X(D) \) is locally free of rank one, or invertible.

We say “imagined” because if the local rings at prime divisors are not DVRs, there is no way to assign an “order of vanishing along each prime divisor,” so no way to write down a formal sum of codimension one closed subschemes of \( X \) associated to \( f \). Instead, we treat the situation more abstractly, and keep track only of \( f \) (up to unit multiple). If \( X \) is not reduced and irreducible, we work with the sheaf of total quotient rings on \( X \).

\[ \text{Remark 6.11.} \] Although we don’t need this level of technical generality here, we can expand on this point of view to define Cartier divisors on more general (not necessarily reduced, irreducible or normal) schemes. Namely we can define a Cartier divisor as the choice of a rational function \( f_\lambda \) (up to unit multiple) on each open set \( U_\lambda \) on some cover of \( X \) so that they agree (up to unit multiple) on the overlaps. We can imagine that these \( f_\lambda \) define a formal sum of codimension one subschemes on \( U_\lambda \) which patch together to an imagined locally principal Cartier divisor on \( X \). That is, we define a Cartier divisor as a global section of the quotient sheaf \( \mathcal{K}^\times / \mathcal{O}_X^\times \). For concreteness sake, we won’t do this here, but rather we stick to the main cases of a smooth (or more generally, normal) variety. See [Har77][II §6].
is a generator (this is well-defined up to unit multiple in $\mathcal{O}_X^\times(U_\lambda)$). Define a Cartier divisor $D$ whose local defining equation on $U_\lambda$ is $g_\lambda^{-1}$ (that is, so that $D \cap U_\lambda = \text{div}_{U_\lambda} g_\lambda^{-1}$). This is well-defined, since on $U_\lambda \cap U_{\lambda'}$, the generators $g_\lambda$ and $g_{\lambda'}$ satisfy $g_\lambda^{-1} g_{\lambda'} \in \mathcal{O}_X^\times(U_\lambda \cap U_{\lambda'})$. It is easy to verify that $\mathcal{O}_X(D) = \mathcal{L}$ as subsheaves of $\mathcal{K}$. Thus every invertible subsheaf of $\mathcal{K}$ is isomorphic to $\mathcal{O}_X(D)$ for some Cartier divisor $D$.

Finally, to see that every invertible sheaf $\mathcal{L}$ is isomorphic to a subsheaf of $\mathcal{K}$, simply tensor the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{K}$ with $\mathcal{L}$. The resulting inclusion $\mathcal{L} \hookrightarrow \mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$ identifies $\mathcal{L}$ with a subsheaf of $\mathcal{K}$, and hence with some $\mathcal{O}_X(D)$ as above. \hfill \square

**Remark 6.14.** Proposition [6.13] shows that there is a one-one correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}$. The invertible sheaves are isomorphic if and only if the corresponding divisors are linearly equivalent (Definition 6.4).

**Remark 6.15.** More generally, on a normal variety $X$, there is a one-to-one correspondence between arbitrary Weil divisors $D$ and rank one reflexive subsheaves $\mathcal{O}_X(D)$ of $\mathcal{K}$. Again, the sheaves are isomorphic if and only if the divisors are linearly equivalent. We won’t need this right now.

For a normal variety $X$, the smooth locus $X^{sm}$ is an open set whose complement has codimension two or more (Proposition [4.17] c). So there is a bijection between divisors on $X$ and divisors on the smooth locus $X^{sm}$ given by simply restricting to $X^{sm}$ — the prime divisors are “too big” (dimension $\dim X - 1$) to “fit” into $X \setminus X^{sm}$ (dimension at most $\dim X - 2$).

Because all divisors on a smooth variety are Cartier, an arbitrary divisor $D$ of $X$ will be Cartier over $X^{sm}$. Thus, on a normal variety, we can think of Weil divisors as the closures (in $X$) of Cartier divisors on the smooth locus $X^{sm}$. Likewise, we can think of rank one reflexive sheaves as the unique extensions to $X$ of invertible sheaves on smooth locus.

**6.1.2. Divisors of zeros of invertible sheaves.** Let $\mathcal{L}$ be an invertible sheaf on a normal variety $X$. By definition, this means that $\mathcal{L}$ has a trivialization—that is, an affine cover $U_\lambda$ of $X$ such that each $\mathcal{L}|_{U_\lambda}$ is free of rank one, say generated by $g_\lambda \in \mathcal{L}(U_\lambda)$. Note that each $g_\lambda$ is uniquely defined by unit multiple on $U_\lambda$, that is, if $g_\lambda$ and $g_{\lambda'}$ are both generators on $U_\lambda$, then there is a $u_\lambda \in \mathcal{O}_X^\times(U_\lambda)$ such that $f g_\lambda = u_\lambda g_{\lambda'}$.

**Definition 6.16.** Let $s$ be a global section of an invertible sheaf $\mathcal{L}$ on a normal irreducible Noetherian scheme $X$. The **divisor of zeros** is the effective divisor defined by

$$\text{Div}_{U_\lambda} f_\lambda \text{ on } U_\lambda$$

\[10\text{If } X \text{ is not normal or irreducible, this still holds using the point of view on divisors described in Remark 6.11.}\]
where
\[ L_{|U_\lambda} = g_\lambda \mathcal{O}_{X|U_\lambda} \quad s_{|U_\lambda} = g_\lambda f_\lambda \]
on a trivialization.

**Example 6.17.** The divisor of zeros of a global section \( s \) of the invertible sheaf \( \mathcal{O}_X(D) \) on a normal Noetherian scheme \( X \) agrees with the effective divisor \( \text{div} \, s + D \) we have already associated to \( s \) in Remark 6.5.

For example, if \( D \) is an effective divisor, then \( 1 \in \mathbb{K} \) is a global section of \( \mathcal{O}_X(D) \). Its divisor of zeros simply recovers \( D \). To see this, say on a sufficiently small affine set \( U \), \( D \) is locally defined by the regular function \( c \in \mathcal{O}_X(U) \); that is, \( D \cap U = \text{div}_U(c) \). But also on \( U \), \( \mathcal{O}_X(D)|_U \) is the free \( \mathcal{O}_U \)-module generated by \( 1 \). Thus the global section \( 1 \in \mathcal{O}_X(D)(X) \) restricts to the function \( 1 = c1 \), and its divisor of zeros on \( U \) is therefore \( \text{div}_U(c) = D \cap U \). So the divisor of zeros of the global section \( 1 \) of \( \mathcal{O}_X(D) \) is the effective Cartier divisor \( D \).

**Remark 6.18.** An invertible sheaf \( L \) on \( X \) is sometimes called a “line bundle.” This is because every invertible sheaf is isomorphic to the sheaf of sections of an actual line bundle \( L \to X \). The word “line bundle” is also used for the linear equivalence class of (Cartier) divisors \( D \) such that \( L \cong \mathcal{O}_X(D) \), or even for a specific representatives of that class. Use caution! Divisors \( D \) are better understood as corresponding to sections of a line bundle \( L \to X \), with the linearly equivalence class of divisors corresponding to isomorphism classes of line bundles.

### 6.2. Global F-regularity

We are now ready to define global F-regularity. We use the approach of [SS10] though this was first defined in [Smi00].

It is easy to see that \( \mathcal{O}_X \subset \mathcal{O}_X(D) \) for any effective divisor \( D \). Thus for all \( e \in \mathbb{N} \), we have an inclusion of \( \mathcal{O}_X \)-modules \( F^e_* \mathcal{O}_X \subset F^e_* \mathcal{O}_X(D) \). If \( X \) is Frobenius split, we can ask whether a splitting
\[ F^e_* \mathcal{O}_X \to \mathcal{O}_X \]
extends to the larger \( \mathcal{O}_X \)-module \( F^e_* \mathcal{O}_X(D) \), that is, whether the natural composition map
\[ \mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X(D) \]
splits as a map of \( \mathcal{O}_X \)-modules.

**Definition 6.19.** Let \( X \) be an F-finite scheme, and let \( D \) be an effective (Cartier) divisor on \( X \). We say that \( X \) is **eventually Frobenius split along** \( D \) if there exists \( e \in \mathbb{N} \) such that the composition
\[ \mathcal{O}_X \to F^e_* \mathcal{O}_X \hookrightarrow F^e_* \mathcal{O}_X(D) \]
\[ \text{or, if } X \text{ is integral and normal, we can take } D \text{ to be an arbitrary effective Weil divisor} \]
splits as a map of \( \mathcal{O}_X \)-modules. We say that \( X \) is **globally F-regular** if, for all effective Cartier divisors \( D \), \( X \) is eventually Frobenius split along \( D \).

A splitting along \( D \) is a map \( \phi \in \text{Hom}_X(F^e_*(\mathcal{O}_X(D)), \mathcal{O}_X) \) such that \( \phi(F^e_*(1)) = 1 \), where by \( F^e_*(1) \in F^e_*(\mathcal{O}_X(D)) \) we mean the natural global section \( F^e_*(1) \in F^e_*K \). Here, \( 1 \in \mathcal{O}_X(D) \) is a global section whose divisor of zeros is exactly \( D \).

Globally F-regular varieties are Frobenius split in a strong sense: there are typically many splittings of Frobenius. Indeed, consider any effective divisor \( D \) on a globally F-regular variety \( X \). A splitting along \( D \) is a map \( \phi \in \text{Hom}_X(F^e_*(\mathcal{O}_X(D)), \mathcal{O}_X) \) such that \( \phi(F^e_*(1)) = 1 \), where by \( F^e_*(1) \) we mean the natural global section \( F^e_*(1) \). Here, \( 1 \in \mathcal{O}_X(D) \) is a global section whose divisor of zeros is exactly \( D \).

**Remark 6.20.** We wish to reconcile the Definition of global F-regularity with the for rings, Definition 6.19. Observe that if \( c \) is a non-zero divisor in a ring \( R \), then there is an isomorphism of rank one free \( R \)-modules \( cR \to R \) given by multiplication by \( c \). Hence in characteristic \( p \), there is also an isomorphism of \( R \)-modules \( F^e_*(cR) \to F^e_*R \) given by multiplication by \( F^e_*(c) \) for all \( e \). Thus to split the map

\[
R \to F^e_*(cR) \quad 1 \mapsto F^e_*(c)
\]

is entirely equivalent to splitting the map

\[
(6.20.1) \quad R \to F^e_*(\frac{1}{c}R) \quad 1 \mapsto F^e_*(1).
\]

Indeed, if \( \phi \in \text{Hom}_R(F^e_*(\frac{1}{c}R), R) \) sends \( F^e_*(1) \mapsto 1 \), then premultiplication by \( F^e_*(c) \) produces \( \psi = \phi \circ F^e_*(\frac{1}{c}) \in \text{Hom}_R(F^e_*R, R) \) sending \( F^e_*(c) \mapsto 1 \). For a Cartier divisor \( D \), the map in (6.20.1) is the local picture of the map

\[
\mathcal{O}_X \longrightarrow F^e_*(\mathcal{O}_X(D)), \quad 1 \mapsto F^e_*(1)
\]

appearing in Definition 6.19 where \( c \) is a local defining equation for the effective Cartier divisor \( D \) Definition (6.9).

**Proposition 6.21.** For a Noetherian irreducible affine scheme \( X = \text{Spec} R \), global F-regularity and local F-regularity are equivalent.

**Proof.** Assume that \( R \) is F-regular. Let \( D \) be a Cartier divisor on \( X = \text{Spec} R \). The submodule \( M = \mathcal{O}_X(-D)(X) \) of the fraction field of \( R \) is finitely generated. So there exists a non-zero-divisor sending \( 1 \mapsto F^e_*(1) \) splits. Restricting to \( F^e_*M \), we have a splitting of \( \mathcal{O}_X \to F^e_*(\mathcal{O}_X(-D)) \), and can conclude that \( X \) is globally F-regular. Conversely, assume \( X = \text{Spec} R \) is globally F-regular. To show that \( R \) is F-regular, take any non-zero-divisor \( c \). It defines a Cartier divisor \( D \) on \( X = \text{Spec} R \), so we know the appropriate map splits because \( \text{Spec} R \) is eventually Frobenius split along \( D \). \( \square \)
**Proposition 6.22.** If an irreducible Noetherian scheme $X$ is globally $F$-regular, then every local ring $\mathcal{O}_{X,x}$ is $F$-regular, that is, it is locally $F$-regular.

**Proof.** Pick any non-zero $c \in \mathcal{O}_{X,x} = R$. We need to find $e$ such that the map

$$R \to F^e_s \left( \frac{1}{c} R \right) \quad 1 \mapsto F^e_s 1$$

splits (Remark [6.20]). Viewing $c$ as a rational function on $X$, we note that $c$ is regular on some neighborhood $U$ of $x$, and hence defines an effective divisor $D_U$ on $U$. Extend $D$ arbitrary to an effective divisor on $X$ (for example, by simply taking the closure, in $X$ of each of the prime divisors on $U$ making up $D_U$). On $U$, we have that $D_U = \text{Div}_U c$ and $\mathcal{O}_X(D)_U$ is generated by $\frac{1}{c}$ as an $\mathcal{O}_U$-module.

Because $X$ is globally $F$-regular, there is an $e$ such that the map

$$\mathcal{O}_X \to F^e \mathcal{O}_X(D) \quad 1 \mapsto F^e_s 1$$

splits. Therefore, the stalk at $x$,

$$R \to F^e_s \left( \frac{1}{c} R \right) \quad 1 \mapsto F^e_s 1$$

also splits. [NEED TO ADD REMARK: $D$ may not be Cartier. FIX later.] \(\square\)

**Remark 6.23.** Global $F$-regularity can be equivalently defined using the language of abstract invertible sheaves rather than divisors. For all invertible sheaves $\mathcal{L}$ on an $F$-finite integral scheme $X$, and all non-zero global sections $c \in \mathcal{L}(X)$, there is an $\mathcal{O}_X$-module map $\mathcal{O}_X \to \mathcal{L}$ sending $1 \mapsto c$ on each open set. Therefore, there is a map

$$F^e_s \mathcal{O}_X \to F^e_s \mathcal{L}$$

given by restricting scalars, and hence a composition

$$\mathcal{O}_X \to F^e_s \mathcal{O}_X \to F^e_s \mathcal{L} \quad 1 \mapsto F^e_s 1 \mapsto F^e_s c.$$  

The scheme $X$ is **globally $F$-regular** if for every choice of $\mathcal{L}$ and every choice of non-zero global section $c \in \mathcal{L}(X)$, there is an $e \in \mathbb{N}$ such that the map

$$F^e_s \mathcal{O}_X \to F^e_s \mathcal{L} \quad 1 \mapsto F^e_s c$$

splits as a map of $\mathcal{O}_X$-modules.

This is equivalent to Definition [6.19] because effective Cartier divisors correspondence to global sections of invertible sheaves (up to unit multiple). For a non-zero global section $c$ of an invertible sheaf $\mathcal{L}$, if $D$ is the divisor of zeros of $c$, then the map $\mathcal{O}_X \to \mathcal{O}_X(D)$ sending $1 \mapsto 1$ is equivalent to the the map $\mathcal{O}_X \to \mathcal{L}$ sending $1 \mapsto c$ (up to multiplying $c$ by a unit in $\mathcal{O}_X(X)$, which does not effective splitting (Remark [4.11])).
6.3. Vanishing Theorems for Globally F-regular Varieties. Globally F-regular varieties satisfy even stronger vanishing theorems than Frobenius split varieties. Recall that an invertible sheaf $L$ is **nef** if the pull back of $L$ has non-negative degree on every curve contained in $X$. (See §1.4 of [Laz04] for a detailed discussion of nefness.)

Equivalently, on normal $X$, we say that a Cartier divisor $D$ is nef if the sheaf $\mathcal{O}_X(D)$ is nef. In the language of intersection theory, we say that $D$ is nef if and only if $D \cdot C \geq 0$ for all curves $C$ in $X$. All ample invertible sheaves are nef, as is the trivial sheaf.

Like ampleness, clearly, a Cartier divisor $D$ is nef if and only if $nD$ is nef for any (equivalently) some $n \in \mathbb{N}$. A nef divisor is “close to ample” in the sense that if $D$ is nef, then for all $n \in \mathbb{N}$, $nD + H$ is ample, for any ample $H$.

**Theorem 6.24.** If $X$ is a globally F-regular variety, then $H^i(X, L) = 0$ for any nef invertible sheaf $L$. In particular, $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.

**Corollary 6.25.** A hypersurface of degree $n + 1$ in $\mathbb{P}^n$ is never globally F-regular. In particular, elliptic curves are not globally F-regular.

The corollary immediate because for such a hypersurface $X$, we have $\omega_X = \mathcal{O}_X$ and $H^n(X, \omega_X) \cong k$ is never zero for an $n$-dimensional projective variety $X$. The statement for elliptic curves follows by realizing the elliptic curve as a curve of degree 3 in $\mathbb{P}^2$.

**Remark 6.26.** Recall that for $d > n + 1$, a hypersurface of degree $d$ in $\mathbb{P}^n$ is never globally F-regular, because it is not Frobenius split (Example 5.3). So for $d \geq n + 1$, a hypersurface of degree $d$ in $\mathbb{P}^n$ is never globally F-regular. We will prove later than there are many hypersurfaces of degree less than $n$ in $\mathbb{P}^n$ that are globally F-regular. Of course since globally F-regular implies locally F-regular, it can not be the case that every such hypersurface is globally F-regular...F-regularity is much more subtle than that. However, we expect a smooth hypersurface of degree less than $n$ in $\mathbb{P}^n$ to be globally F-regular, provided nothing funny is happening for small characteristics (see [SS10]).

Theorem 6.24 will follow easily from the following more general vanishing theorem:

**Theorem 6.27.** Suppose that a Noetherian scheme $X$ is eventually Frobenius split along some effective Cartier divisor $D$. If $L$ is an invertible sheaf on $X$ such that $H^i(X, L^n(D)) = 0$ for $n \gg 0$, then $H^i(X, L) = 0$. In particular, if $X$ is globally F-regular, then for any invertible $L$ for which there exists an effective Cartier divisor $D$ such that $H^i(X, L^n(D)) = 0$ for $n \gg 0$, then we have $H^i(X, L) = 0$.

\[\text{Put differently, a divisor } D \text{ is nef if and only if for all small } \varepsilon > 0, \text{ the divisor } D + \varepsilon H \text{ is ample—adding a tiny ample to a nef divisor makes it ample. This can be made rigorous by using the ample cone inside the Neron-Severi group: the nef divisors are those in the closure of the ample cone. See } [\text{Laz04}].\]
Proof. Because $X$ is eventually Frobenius split along $D$, the map
\[ \mathcal{O}_X \to F^e_\ast \mathcal{O}_X(D) \quad 1 \mapsto F^e_\ast 1 \]
splits for all large $e$. Now tensoring with the invertible sheaf $\mathcal{L}$, the induced map
\[ \mathcal{L} \to F^e_\ast \mathcal{O}_X(D) \otimes \mathcal{L} \cong F^e_\ast (\mathcal{O}_X(D) \otimes F^e_\ast \mathcal{L}) \cong F^e_\ast (\mathcal{O}_X(D) \otimes \mathcal{L}^p) = F^e_\ast (\mathcal{L}^p(D)) \]
splits, where the first isomorphism is obtained from the projection formula. This induces a split inclusion of cohomology groups
\[ H^i(X, \mathcal{L}) \hookrightarrow H^i(X, F^e_\ast (\mathcal{L}^p(D))) \cong H^i(X, \mathcal{L}^p(D)) \]
with the isomorphism holding because the Frobenius map $F^e$ is affine. Now it is immediate that if $H^i(X, \mathcal{L}^p(D))$ vanishes for any $e \gg 0$, then $H^i(X, \mathcal{L})$ vanishes as well. \qed

Proof of Corollary 6.24. Suppose $\mathcal{L}$ is nef. Fix an ample effective divisor $H$. Then $\mathcal{L}^n(H)$ is ample for all $n \geq 0$. Since $X$ is Frobenius split, we know that cohomology vanishes for an ample invertible sheaf, so $H^i(X, \mathcal{L}^n(H)) = 0$ for all $n \geq 0$ and all $i \geq 1$. Now it follows from Theorem 6.27 that $H^i(X, \mathcal{L}) = 0$ for all $i \geq 1$. \qed

We also get the following form of the Kawamata–Viehweg Vanishing theorem for globally F-regular varieties.

Corollary 6.28. Let $X$ be a globally F-regular projective variety and let $\mathcal{L}$ be a big and nef invertible sheaf on $X$. Then $H^i(X, \omega \otimes \mathcal{L}) = 0$ for all $i > 0$.

Proof. Since $X$ is globally F-regular, it is locally F-regular, hence Cohen-Macaulay, so we can use Serre Duality. By Serre duality [Har77][III Thm 7.6], it suffices to show $H^i(X, \mathcal{L}^{-1}) = 0$ for all $i < \dim X$.

Because $\mathcal{L}$ is big and nef, we can find an effective Cartier divisor $D$ such that $\mathcal{L}^m(-D)$ is ample for all $m \gg 0$ [Laz04][Ex 2.2.19]. But then $H^i(X, \omega_X \otimes (\mathcal{L}^m(-D))^n) = 0$ for all $i > 0$ and all $n \gg 0$ by Serre Vanishing. So by Serre Duality, again, $H^i(X, (\mathcal{L}^{-m}(D))^n) = 0$ for all large $n$ and all $V < \dim X$. Since $X$ is Frobenius Split, Theorem 5.4 implies that $H^i(X, \mathcal{L}^{-m}(D)) = 0$ for all $i < \dim X$. Now by Theorem 6.27, we conclude that $H^i(X, \mathcal{L}^{-1}) = 0$. This proves Kawamata Viehweg Vanishing on any globally F-regular projective variety. \qed

7. Frobenius Splitting for Projective Varieties and their Affine Cones

We have proved beautiful vanishing theorems for projective varieties with Frobenius splitting but we still have not seen a single example of a projective variety that is Frobenius split or globally F-regular!

We remedy this now by proving that a projective variety $X$ is Frobenius split (or globally F-regular) if its homogeneous coordinate ring is.
Corollary 7.1. Let $X \subset \mathbb{P}^n_k$ be a projective variety with homogeneous coordinate ring $R = \frac{k[x_0, \ldots, x_n]}{I_X}$. If $R$ is Frobenius split (respectively, F-regular), then $X$ is Frobenius split (respectively, globally F-regular).

For example, it follows immediately that projective space $\mathbb{P}^n$ is globally F-regular. In particular, a genus zero curve $\mathbb{P}^1$ is globally F-regular in every characteristic.

As another example, any quotient of $\mathbb{P}^n$ by a finite group $G$ (whose order is not divisible by the characteristic $p$) is globally F-regular. This is because the homogeneous coordinate ring of $\mathbb{P}^n/G$ is the ring of invariants $k[x_0, \ldots, x_n]^G$, which we have checked to be F-regular in Proposition 4.6 (see also Example 2.6).

The corollary is a special case of the following more general theorem.

Theorem 7.2. Let $S$ be an $\mathbb{N}$-graded F-finite ring (of characteristic $p$). If $S$ is Frobenius split, then so is the projective scheme Proj $S$. Likewise, if $S$ is finitely generated and F-regular, then so is Proj $S$.

We will prove Theorem 7.2 and its corollary after reviewing the “Proj” construction on a graded ring in §7.1. Then, in Section 7.2 we will examine the converse. Remember that many different graded rings $S$ can determine the same projective scheme $X = \text{Proj } S$. It is not the case that if $X$ is globally F-regular, then every graded ring $S$ determining $X$ must be F-regular. However, as we will prove in Theorem 7.7, global F-regularity (or Frobenius splitting) of a projective variety Proj $S$ is equivalent to global F-regularity (or Frobenius splitting) of Spec $S$ when the graded ring $S$ is a section ring for $X$.

7.1. From graded rings to projective schemes. Let $S$ be an $\mathbb{N}$-graded ring. The $\mathbb{N}$-grading means that, as an abelian group,

$$S = \bigoplus_{n \in \mathbb{N}} S_n$$

and that for all natural numbers $n, m$, $S_n S_m \subset S_{n+m}$. In particular, $S_0$ is a subring.

[If we say that $S$ is a “finitely generated $\mathbb{N}$-graded ring”, we mean that $S$ is finitely generated as an algebra over this subring $S_0$.]

Let $S_+$ denote the irrelevant ideal of $S$, namely the ideal $S_+ = \bigoplus_{n>0} S_n$. The ideal $S_+$ is homogeneous. By definition, an ideal $I$ of $S$ is homogeneous if whenever $f = \bigoplus f_n \in I$, then each homogenous component $f_n \in I$. Equivalently, $I$ is homogeneous if it can be generated by homogenous elements.

13Again, the word “curve” in this context means smooth projective variety of dimension one.
7.1.1. The Scheme $\text{Proj}(S)$. The scheme $\text{Proj} S$ is defined as follows.

- As a set, $\text{Proj} S$ is the set of homogeneous prime ideals of $S$ that do not contain the irrelevant ideal. That is, $\text{Proj} S$ consists of the homogeneous primes in $\text{Spec} S \setminus \mathbb{V}(S_+)$. 

- As a topological space, we endow $\text{Proj} S$ with the (subspace) topology inherited from the Zariski topology on $\text{Spec} S$. A basis for this topology consists of the open sets of the form $D_+(f)$ where $f$ is a homogeneous element of $S_+$, where

$$D_+(f) = \{P \in \text{Proj} S \mid f \notin P\}.$$ 

The chart $D_+(f)$ is homeomorphic to the prime spectrum of the ring $\left[S_+^{\frac{1}{f}}\right]_0$ of all degree zero elements in the localization $S_+^{\frac{1}{f}}$.

- The sheaf of rings $\mathcal{O}_X$ on $X = \text{Proj} S$ is defined on the basis by

$$\mathcal{O}_X(D_+(f)) = \left[S_+^{\frac{1}{f}}\right]_0.$$ 

One can check that if $D_+(g) \subset D_+(f)$, then there is a map of rings

$$\left[S_+^{\frac{1}{f}}\right]_0 \to \left[S_+^{\frac{1}{g}}\right]_0$$

which defines a restriction map for the sheaf. This uniquely defines the scheme structure on $\text{Proj} S$: it has a basis of affine open subschemes of the form $D_+(f) = \text{Spec} \left[S_+^{\frac{1}{f}}\right]_0$, where $f$ ranges through the homogeneous elements in $S$ of positive degree.

For every homogeneous $f \in S$ of positive degree, there is a natural ring map $S_0 \to \left[S_+^{\frac{1}{f}}\right]_0$ induced by the inclusion $S_0 \subset S$. This means that each $D_+(f)$ is a scheme over $\text{Spec} S_0$, and that there is a map of schemes $\text{Proj} S \to \text{Spec} S_0$. If $S$ is finitely generated over $S_0$, then each ring $\left[S_+^{\frac{1}{f}}\right]_0$ is finitely generated over the subring $S_0$ and the map $\text{Proj} S \to \text{Spec} S_0$ is of finite type. In this case, $\text{Proj} S$ is a projective scheme over $\text{Spec} S_0$.

7.1.2. The Quasicoherent sheaf of a graded module. Let $M$ be a $\mathbb{Z}$-graded $S$-module. By definition, $M$ admits a decomposition (as an abelian group) indexed by $\mathbb{Z}$

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

such that whenever $s \in S_n$ and $m \in M_d$, the product $sm$ is in $M_{n+d}$.

Every $\mathbb{Z}$-graded $S$-module $M$ determines a unique quasicoherent sheaf $\widetilde{M}$ on $\text{Proj} S$ defined by

$$\widetilde{M}(D_+(f)) = \left[M_+^{\frac{1}{f}}\right]_0.$$
This is obviously a module over the ring $O_X(D_+(f)) = [S[\frac{1}{p}]_0]$. In particular, the sheaf $\tilde{S}$ recovers the structure sheaf $O_X$. Every degree preserving homomorphism of graded $S$-modules, $M \to N$, induces a corresponding map of quasi-coherent sheaves on $\text{Proj} S$, $\tilde{M} \to \tilde{N}$. That is, there is a functor between categories

\[
\begin{align*}
\{\text{Z-graded } S\text{-modules, with their degree preserving homomorphisms}\} \\
\downarrow
\{\text{quasi-coherent sheaves on } \text{Proj} S \text{ and their homomorphisms}\}.
\end{align*}
\]

The sheaf $\tilde{M}$ is coherent if $M$ is finitely generated over $S$ and $S$ is Noetherian.

**Example 7.3.** An important special case is when $A = k$ is a field and $S$ is finitely generated. In this case, $S_+$ is the unique homogeneous maximal ideal $m$ of $S$ and $\text{Proj} S$ is a projective scheme over $k$. If, furthermore, $S$ is finitely generated over $k$ by its elements of degree one, then $\text{Proj} S \cong \mathbb{V}(f_1, \ldots, f_t) \subset \mathbb{P}^n_k$ where $S \cong k[x_0, \ldots, x_n] / \langle f_1(x_1, \ldots, x_n), \ldots, f_t(1, x_1, \ldots, x_n) \rangle$ for some homogeneous polynomials $f_1, \ldots, f_t$. The affine chart

\[
D_+(x_0) = \text{Spec} \left[ R[\frac{1}{x_0}]_0 \right] \cong k[\frac{x_1}{x_0}, \ldots, \frac{x_1}{x_0}] / \langle f_1(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}), \ldots, f_t(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \rangle
\]

and similarly for the other basic open affines $D_+(x_i)$ in the standard cover of $\mathbb{P}^n$.

**7.1.3. A worked example.** Before proving Theorem 7.2, it is instructive to work out an example: namely, we show that projective space $\mathbb{P}^n_F$ is Frobenius split. The splitting is induced in a natural way from the splitting of Frobenius on its homogeneous coordinate ring $R = F_p[x_0, \ldots, x_n]$. This sounds straightforward, and mostly is, but there are some subtleties we work through in detail here.

We have already seen that $(\mathbb{F}_p[x_0, \ldots, x_n]^{1/p})$ is a free module over $\mathbb{F}_p[x_0, \ldots, x_n]$ on the basis $\{x_0^{a_1} \cdots x_n^{a_n}\}_{0 \leq a_i < p}$. The projection onto the summand spanned by the basis element 1 is an $R$-linear map

\[
\mathbb{F}_p[x_0^{1/p}, \ldots, x_n^{1/p}] \to \mathbb{F}_p[x_0, \ldots, x_n]
\]

which gives an $R$-module splitting of the Frobenius map $R \hookrightarrow R^{1/p}$. This is a map of graded $R$-modules if we grade $F_1R$ in the obvious way, that is, by assigning the degree $\frac{1}{p}$ to each of the variables $x_i^{1/p}$.

We would like conclude, by looking at the corresponding map of sheaves on $\text{Proj} R = \mathbb{P}^n$, that we have induced a splitting of Frobenius. However, there are two obvious problems: First, the module $\mathbb{F}_p[x_0^{1/p}, \ldots, x_n^{1/p}]$ is not $\mathbb{Z}$-graded, but rather $\frac{1}{p}\mathbb{Z}$-graded. Worse, in affine charts, $F_*O_{\mathbb{P}^n}$ should have rank $p^n$ over $O_{\mathbb{P}^n}$, not rank $p^{n+1}$ like $R^{1/p}$ over $R$. 
Both these problems can be taken care of as follows. Consider the graded $R$-submodule $[R^{1/p}]_Z$ of $R^{1/p}$ consisting of elements of integral degrees. Note that this is a subring of $R^{1/p}$ containing $R$, but it is strictly larger than $R$, as it includes elements like $(x_1x_2^{p-1})^{1/p}$, which has degree one.

Simply by restricting $\pi$ to the subring $[R^{1/p}]_Z$, we clearly have a splitting of the inclusion of $R$ into $[R^{1/p}]_Z$, that is the composition of $R$-modules

$$R \hookrightarrow [R^{1/p}]_Z \xrightarrow{\pi} R$$

is the identity map. Thus the corresponding map of sheaves

$$\tilde{R} \hookrightarrow \widetilde{[R^{1/p}]_Z} \xrightarrow{\tilde{\pi}} \widetilde{R}$$

is the identity map as well. We claim that the map of coherent sheaves $\tilde{R} \rightarrow \widetilde{[R^{1/p}]_Z}$ is the Frobenius map $O_X \rightarrow \mathcal{F}_*O_X$, in which case it follows that $\tilde{\pi}$ is a splitting of Frobenius for $\mathbb{P}^n$.

To verify our claim, we investigate the sheaf $\widetilde{[R^{1/p}]_Z}$ on $\mathbb{P}^n$ by computing its sections on the affine chart $D_+(x_0)$. Since $R^{1/p}$ is freely generated over $R$ by

$$\{(x_0^{a_0}x_1^{a_1} \cdots x_n^{a_n})^{1/p}\}_{0 \leq a_i < p},$$

these same generators induce a free basis after localization at $x_0$. However, only those monomials where $a_0 + a_1 + \cdots + a_n$ is divisible by $p$ are in $[R^{1/p}]_Z$. The degree zero part of $[R^{1/p}]_Z[\frac{1}{x_0}]$ is thus spanned over $\mathbb{F}_p$ by elements

$$\frac{(x_0^{a_0}x_1^{a_1} \cdots x_n^{a_n})^{1/p}}{x_0^d},$$

where $a_0 + \cdots + a_n = pd$, and it is spanned over the subring $[R[\frac{1}{x_0}]_0]$ by these same elements, restricted to the range $0 \leq a_i < p$. These generators can be rewritten as

$$[\left(\frac{x_1}{x_0}\right)^{a_1} \cdots \left(\frac{x_n}{x_0}\right)^{a_n}]^{1/p},$$

showing that the monomials in $(\frac{x_1}{x_0})^{1/p}, \ldots, (\frac{x_n}{x_0})^{1/p}$ are a basis over $\mathbb{F}_p$, with those in which $(\frac{x_1}{x_0})^{1/p}$ appears with exponent at most $p - 1$ a basis over the subring $\mathbb{F}_p[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$. This is exactly the standard free basis we have earlier described for $(\mathbb{F}_p[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}])^{1/p}$ over the polynomial ring $\mathbb{F}_p[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$. That is, these elements are a free basis for $\widetilde{[R^{1/p}]_Z}(D_+(x_0))$ over $O_{\mathbb{P}^n}(D_+(x_0)) = \mathbb{F}_p[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$. Likewise the map induced by the inclusion $R \hookrightarrow [R^{1/p}]_Z$ is the Frobenius inclusion $\mathbb{F}_p[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] \hookrightarrow (\mathbb{F}_p[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}])^{1/p}$. So on the basic open affine $D_+(x_0)$, the map $\tilde{\pi}$ is indeed the Frobenius map on $D_+(x_0)$.

Of course, there is nothing special here about $x_0$: a similar calculation works in each of the basic affine charts $D_+(x_i)$ in the standard affine cover of $\mathbb{P}^n$. That is, the graded $R$-module $[R^{1/p}]_Z$ determines the sheaf $\mathcal{F}_*O_{\mathbb{P}^n}$ on $\mathbb{P}^n_{\mathbb{F}_p}$ and the map $\tilde{\pi}$ is
a splitting of the Frobenius map $\mathcal{O}_{\mathbb{P}^n} \to F_*\mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$. We conclude that $\mathbb{P}^n_{\mathbb{F}_p}$ is a Frobenius split scheme for every $n$.

The proof of Theorem 7.2 is essentially the same as our calculation for $\mathbb{P}^n$. To make this clear, we isolate an important idea from our calculation.

7.1.4. Frobenius and Graded Rings. Let $S$ be an arbitrary $\mathbb{N}$-graded ring of characteristic $p$, and let $X = \text{Proj} S$. We know that $\mathcal{O}_X$ is the sheaf $\tilde{S}$. But is there a graded $S$-module that determines $F_*\mathcal{O}_X$? It is tempting to guess that $F_*S$ might determine $F_*\mathcal{O}_X$, but as we saw in the case of projective space, this doesn’t quite work.

Let $M$ be any $\mathbb{Z}$-graded $S$-module. There is a natural way to grade $F^e_*M$ by $\frac{1}{p^e}\mathbb{Z}$. Specifically, if $m \in M$ is homogeneous, then viewed as an element of $F^e_*M$, we define
$$\deg F^e_*m = \frac{1}{p^e}\deg m.$$ With this grading, the Frobenius map
$$S \to F^e_*S \quad s \mapsto F^e_*s^p$$ is a degree-preserving map of rings. Furthermore, the natural action of $S$ on $F^e_*M$ respects the degrees: $s \in S_i$ acts on $F^e_*m \in [F^e_*M]_{\frac{i}{p^e}}$ to produce $sF^e_*(m) = F^e_*(s^{p^e}m) \in [F^e_*M]_{\frac{i}{p^e} + i}$.

The module $F^e_*M$ is not $\mathbb{Z}$-graded! Rather $F^e_*M$ is $\frac{1}{p^e}\mathbb{Z}$-graded. However, it has a natural $\mathbb{Z}$-graded submodule
$$[F^e_*M]_{\mathbb{Z}} = \{F^e_*m \mid \deg F^e_*m \in \mathbb{Z}\} = \{F^e_*m \mid p^e \text{ divides } \deg m\} \subset F^e_*M$$ consisting of the elements in $[F^e_*M]$ that happen to have integral degree. Note that
$$[F^e_*M]_{\mathbb{Z}} = F^e_*M^{(p^e)}$$ where $M^{(p^e)}$ denotes the “Veronese submodule” $M^{(p^e)} = \oplus_{n \in \mathbb{Z}} M_{p^en} \subset M$ of elements of $M$ whose degrees are divisible by $p^e$. Since the elements of $S$ have integer degree, the usual action of $S$ by multiplication on elements of $m$ induces a natural $S$-module action on $[F^e_*M]_{\mathbb{Z}} = F^e_*M^{(p^e)}$. This action respects the degree.

Lemma 7.4. Let $S$ be an $\mathbb{N}$-graded ring of characteristic $p$ and let $M$ be a $\mathbb{Z}$-graded $S$-module. Let $[F^e_*M]_{\mathbb{Z}}$ be the graded submodule of $F^e_*M$ consisting of elements of integral degrees. Then, as quasicoherent sheaves on $\text{Proj} S$,
$$[\tilde{F^e_*M}]_{\mathbb{Z}} = F^e_*(\tilde{M}).$$ In particular, the sheaf $F^e_*\mathcal{O}_X$ on $X = \text{Proj} S$ is determined by the graded $S$-module $[F^e_*S]_{\mathbb{Z}} = F^e_*S^{(p^e)}.$
Proof of Lemma. We compute the sections of \([F^\ast_\mathbb{E}M]_\mathbb{Z}\) over a basic open affine \(D_+(f) \subset \text{Proj} \, S\):

\[
[F^\ast_\mathbb{E}M]_\mathbb{Z}(D_+(f)) = \left[ [F^\ast_\mathbb{E}M]_\mathbb{Z} \left( \frac{1}{f} \right) \ast \right]_0 = \left[ [F^\ast_\mathbb{E}(M[\frac{1}{f^p}])]_0 = F_\mathbb{E}(M[\frac{1}{f^p}])_0 ,
\]

which is precisely \(F_\mathbb{E}M(D_+(f))\). \(\Box\)

Remark 7.5. An easy but important fact: the module inclusion \([F^\ast_\mathbb{E}M]_\mathbb{Z} \subset F^\ast_\mathbb{E}M\) splits as a map of \([F^\ast_\mathbb{E}S]_\mathbb{Z}\)-modules, and hence \(S\)-modules; this is obvious by considering the direct sum decomposition given by degree. Alternatively, we can argue that because \(M(p^e) \hookrightarrow M\) splits as a map of \(S(p^e)\)-modules, also \(F^\ast_\mathbb{E}(M(p^e)) \hookrightarrow F^\ast_\mathbb{E}(M)\) splits as a map of \(F^\ast_\mathbb{E}(S(p^e))\)-modules (and hence \(S\)-modules via the natural degree preserving ring map \(S \to F^\ast_\mathbb{E}(S(p^e))\)).

Proof of Theorem 7.2. Let \(S\) be a Frobenius split graded ring, and let \(\phi \in \text{Hom}_S(F_\mathbb{E}S, S)\) be a splitting of Frobenius. Because \(F_\mathbb{E}S\) is a finitely generated \(S\)-module, the module \(\text{Hom}_S(F_\mathbb{E}S, S)\) is graded. So we can assume without loss of generality that \(\phi\) is homogeneous, and in fact, degree preserving, since \(\phi(F_\ast_1) = 1\).

As in the case of projective space, we can restrict \(\phi\) to the submodule \([F_\ast S]_\mathbb{Z}\) of elements whose degrees are integers. Now, the composition of maps of \(\mathbb{N}\)-graded \(S\)-modules

\[
S \xrightarrow{F_\mathbb{E}} [F_\mathbb{E}S]_\mathbb{Z} \xrightarrow{\phi} S
\]

is the identity map, and so the corresponding map of sheaves on \(\text{Proj} \, S\)

\[
\tilde{S} \xrightarrow{F_\mathbb{E}} \tilde{[F_\mathbb{E}S]_\mathbb{Z}} \xrightarrow{\tilde{\phi}} \tilde{S}
\]

is the identity map as well. Invoking Lemma 7.4, we thus have a splitting of Frobenius on \(\text{Proj} \, S\):

\[
\mathcal{O}_X \xrightarrow{F_\mathbb{E}} F_\mathbb{E}\mathcal{O}_X \xrightarrow{\tilde{\phi}} \mathcal{O}_X.
\]

Thus \(\text{Proj} \, S\) is Frobenius split.

The proof for \(F\)-regularity is similar. First note that because \(S\) is \(F\)-regular, it is normal, and hence a domain (since it is graded-local).14

Given an effective Cartier divisor \(D\), the sheaf \(\mathcal{O}_X(D)\) is a subsheaf of \(\tilde{M}\) for some finitely generated graded \(S\)-module \(M\) contained in the total quotient ring \(K\) of \(S\). To see this, we can choose a basic affine cover which trivializes \(\mathcal{O}_X(D)\), and note that on each chart \(D_+(f_i)\), the sections of \(\mathcal{O}_X(D)\) will be of the form \(\frac{g_i}{h_i} [S[\frac{1}{f_i}]_0\) where \(\frac{g_i}{h_i} \in K\). Our finiteness assumptions on \(S\) implies that \(\mathcal{O}_X(D)\) is trivialized on some

\[14\text{Actually, since we have not assumed that } S_0 \text{ is a field, this is not quite true in the general case, but it is easy to get around this, either by assuming } S_0 \text{ is a domain, or by working with non-zero divisors in the reduced case; we prefer to leave these technicalities as an exercise, since we are mainly interested in the case where } S_0 \text{ is a field.} \]
finite subcover, say given by \( f_1, \ldots, f_t \). Let \( M \) be the submodule of \( K \) generated by \( \frac{g_i}{h_i}, \ldots, \frac{g_t}{h_t} \). One easily checks that \( \mathcal{O}_X(D) \subseteq \widetilde{M} \).

Since \( M \) is finitely generated, there is a non-zero \( c \in S \) such that \( cM \subseteq S \) (for example, we can let \( c \) be the product of the denominators \( h_i \)). This says that

\[
M \subseteq \frac{1}{c} S.
\]

Now, because \( S \) is \( F \)-regular, there exists \( e \in \mathbb{N} \) such that \( S \hookrightarrow F^e \ast (\frac{1}{c} S) \) \( 1 \mapsto F^e 1 \) splits (Remark 6.20). That is, we have a map \( \phi \in \text{Hom}_S(F^e 1, S) \), which can be assumed homogeneous, such that \( \phi(F^e 1) = 1 \).

Restricting \( \phi \) to \( F^e M \), we see that

\[
S \hookrightarrow F^e M \hookrightarrow F^e (\frac{1}{c} S) \xrightarrow{\phi} S
\]

is the identity map of \( S \)-modules, and now restricting to the submodules of integral degree, this remains true:

\[
S \hookrightarrow [F^e M]_Z \hookrightarrow [F^e (\frac{1}{c} S)]_Z \xrightarrow{\phi} S
\]

The corresponding map of sheaves (using Lemma 7.4 to deduce \( [F^e M]_Z = F^e \widetilde{M} \)) gives the identity composition

\[
\mathcal{O}_X \hookrightarrow F^e \widetilde{M} \xrightarrow{\phi} \mathcal{O}_X
\]

(where we somewhat abusively use the notation \( \hat{\phi} \) to denote the restriction of \( \phi \) to the subsheaf \( F^e \widetilde{M} \)). Finally, since \( \mathcal{O}_X(D) \subseteq \widetilde{M} \), this splitting factors as

\[
\mathcal{O}_X \hookrightarrow F^e \mathcal{O}_X(D) \hookrightarrow F^e \widetilde{M} \xrightarrow{\hat{\phi}} \mathcal{O}_X
\]

showing that the desired map \( \mathcal{O}_X \hookrightarrow F^e \mathcal{O}_X(D) \) splits. Since for every effective Cartier divisor \( D \) on \( \text{Proj} S \) we can find such a splitting, we conclude that \( \text{Proj} S \) is globally \( F \)-regular. \( \square \)

7.2. From the Projective Scheme to the Cone. The converse of Theorem 7.2 is false! Indeed, many different rings can determine the same projective scheme, and it is possible that some of those rings are \( F \)-regular or Frobenius split and others are not.

Example 7.6. Let \( X = \mathbb{P}^1 \). Embed \( X \) into \( \mathbb{P}^4 \) using the Veronese mapping; the image curve is isomorphic to \( X \) and has coordinate ring \( S = k[s^4, s^3 t, s^2 t^2, s t^3, t^4] \).

Now project from the point \([0 : 0 : 1 : 0 : 0]\) \( \in \mathbb{P}^4 \) to get a curve in \( \mathbb{P}^3 \) with coordinate ring \( S' = k[s^4, s^3 t, s t^3, t^4] \). The image of \( X \) under the projection is isomorphic to \( X \). Indeed, the inclusion \( S' \hookrightarrow S \) of graded rings induces an isomorphism \( \text{Proj} S' \rightarrow \)
7. FROBENIUS SPLITTING FOR PROJECTIVE VARIETIES AND THEIR AFFINE CONES

Proj \( S \). [On the covering charts \( D_+(s^4) \) and \( D_+(t^4) \), the schemes \( \text{Proj} \, S' \) and \( \text{Proj} \, S \) are identified.]

In this example, we have \( \text{Proj} \, S \cong \text{Proj} \, S' \cong \mathbb{P}^1 \), which is globally F-regular. The ring \( S \) is F-regular, but the ring \( S' \) is not. Indeed, \( S' \) is not even normal: its normalization is \( S \).

More generally, given a projective variety \( X \subset \mathbb{P}^n \), where \( n \) is large relative to the dimension of \( X \), we expect a generic projection from a point not on \( X \) to induce an isomorphism from \( X \) to some variety in \( \mathbb{P}^{n-1} \). The homogeneous coordinate ring for this embedding is never normal (the embedding is not projectively normal). Such a non-normal coordinate ring can never be F-regular, even if \( X \) is globally F-regular.

However, we do have a converse for Theorem 7.2 if we restrict ourselves to a particularly nice, natural ring associated with a projective scheme \( X \) over a field:

**Theorem 7.7.** Let \( X \) be a projective variety of prime characteristic, and let \( \mathcal{L} \) be any ample invertible sheaf on \( X \). Then \( X \) is Frobenius split (respectively, globally F-regular) if and only if the section ring

\[
S(X, \mathcal{L}) := \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}^n)
\]

is Frobenius split (respectively, F-regular).

Before proving Theorem 7.7, we review section rings and affine cones for projective varieties, including Serre’s correspondence between finitely generated modules over the section ring \( S(X, \mathcal{L}) \) and coherent sheaves on \( X \). Sections rings, which we review momentarily, are “the best” (i.e. saturated, \( S_2 \)) graded rings determining a projective scheme \( X \).

### 7.2.1. Section Rings

Given a connected projective variety \( X \) over a field \( k \), we can construct a ring \( S \) so that \( \text{Proj} \, S = X \). The construction depends on the choice of an ample invertible sheaf \( \mathcal{L} \) on \( X \). Given an invertible sheaf \( \mathcal{L} \) on \( X \), define the section ring

\[
S(X, \mathcal{L}) := \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}^n);
\]

this is a graded \( k \)-algebra with a unique homogenous maximal ideal \( m \) generated by the elements of positive degree. The section ring can also be viewed as the global sections of the sheaf of algebras \( \bigoplus_{n \in \mathbb{N}} \mathcal{L}^n \) on \( X \). Section rings need not be finitely generated in general, but they are always finitely generated when \( \mathcal{L} \) is ample. Let us assume henceforth that \( \mathcal{L} \) is ample. By definition, this means that \( \text{Proj} \, S \) recovers \( X \).

**Example 7.8.** In the simplest example, the variety \( X \hookrightarrow \mathbb{P}^n \) is a projectively normal closed subscheme of \( \mathbb{P}^n \), and we can take \( \mathcal{L} \) to be \( \mathcal{O}_X(1) = \iota^* \mathcal{O}_{\mathbb{P}^n}(1) \), the standard hyperplane bundle on \( X \). It is so named because the divisors associated to its
global sections (C.f. §6.5) are the hyperplane sections of \( X \). In this case, the section ring
\[
S = S(X, O_X(1)) := \bigoplus_{n \in \mathbb{N}} H^0(X, O_X(n)),
\]
is precisely the homogeneous coordinate ring \( R = k[x_0, \ldots, x_n]/I_X \) where \( I_X \) is the homogeneous ideal of polynomials vanishing on \( X \). In this case, \( \mathcal{L} \) is very ample.

If \( X \hookrightarrow \mathbb{P}^n_k \) is not normally embedded, the section ring \( S \) built from \( \iota^* \mathcal{O}_{\mathbb{P}^n}(1) \) is almost the homogeneous coordinate ring \( S \): it is a finite integral extension of \( R \) such that \( S_n = R_n \) for all \( n \gg 0 \). \[\text{Har77}][\text{II Ex 5.14}].

### 7.2.2. Serre Correspondence

If \( S = S(X, \mathcal{L}) \) is a section ring for some ample \( \mathcal{L} \) on a projective variety \( X \), then for any coherent sheaf \( M \) on \( X \), we can construct a finitely generated \( \mathbb{Z} \)-graded \( S \)-module
\[
M = \bigoplus_{n \in \mathbb{Z}} H^0(X, M \otimes \mathcal{L}^n)
\]
such that \( \widetilde{M} \) recovers \( M \) on \( X \). This graded \( S \)-module \( M \) is the “largest” (or unique saturated) \( S \)-module which determines \( M \) (any other graded \( S \)-module which eventually agrees with this one in large degree will also determine \( M \)). For example, \( \widetilde{S(d)} \) recovers \( \mathcal{L}^d \). This correspondence between coherent sheaves on \( X \) and finitely generated graded \( S \)-modules (up to “agreement in the tail”) is called \textbf{Serre’s correspondence}. This is covered in \[\text{Har77}][\text{II §5}], but unfortunately there, the rings \( S \) are assumed generated in degree one. Still, most of the proofs are the same for any section ring. \[\text{!!}]

### 7.2.3. The punctured cone over a projective variety

If \( (S, \mathfrak{m}) \) is a section ring, the projective scheme \( \text{Proj} \, S \) usually has much better properties than the cone \( \text{Spec} \, S \), but the differences occur mostly at the “vertex” \( \mathfrak{m} \). For example, if \( \text{Proj} \, S \) is smooth, then \( \text{Spec} \, S \setminus \{\mathfrak{m}\} \) is too, but \( \text{Spec} \, S \) will usually have a singularity at \( \mathfrak{m} \). This is because the punctured cone \( \text{Spec} \, S \setminus \{\mathfrak{m}\} \) is a \( \mathbb{A}^1 \)-bundle over \( \text{Proj} \, S \).

**Proposition 7.9.** Let \( S = S(X, \mathcal{L}) \) be a section ring of an ample invertible sheaf \( \mathcal{L} \) on a connected projective variety \( X \) over \( k \). Then there is a map
\[
\text{Spec} \, S \setminus \{\mathfrak{m}\} \xrightarrow{\pi} X
\]
such that every point \( x \in X \) has a neighborhood \( U \) such that \( \pi^{-1}(U) \cong U \times_{\text{Spec} \, k} \text{Spec} \, k[t, t^{-1}] \).

\[\text{!!}\text{Caution: If } S \text{ is not a section ring, then the sheaves } \widetilde{S(d)} \text{ need not be invertible!}\]

\[\text{!!}\text{At the time of Hartshorne’s writing, it was not clear what the “best” generality would be for the exposition of 1500 page’s of Grothendieck’s EGA.Nearly every statement in II §5 of [Har77] about rings generated in degree one holds also for section rings, with minor adaptations to the proofs. The proofs in full generality can be found in [??].}\]
Before proving this, we deduce a nice geometric corollary.

**Corollary 7.10.** Let \( X \) be a projective variety and \( S = S(X, \mathcal{L}) \) a section ring for some ample \( \mathcal{L} \) on \( X \). The variety \( X = \text{Proj} \ S \) has property \( \mathcal{P} \) if and only if the punctured cone \( \text{Spec} \ S\setminus \{m\} \) has property \( \mathcal{P} \), where \( \mathcal{P} \) can be any of the following properties: reduced, normal, Cohen-Macaulay, locally F-regular, locally Frobenius split, smooth, etc.

**Proof of Corollary 7.10.** As Property \( \mathcal{P} \) can be checked locally, Proposition 7.9 tells us that this corollary amounts to saying the ring \( R \) has property \( \mathcal{P} \) if and only if \( R[t, t^{-1}] \) has property \( \mathcal{P} \). This can be checked directly for each property. Note that \( R \) is a direct summand of \( R[t, t^{-1}] \) (its zeroth graded piece) while \( R[t, t^{-1}] \) is a faithfully flat extension of \( R \). \( \square \)

**Remark 7.11.** When \( S \) is a section ring for a projective variety \( X \), Theorem 7.7 says that the global versions of Frobenius splitting and F-regularity for a projective variety \( X \) amount to the corresponding local properties at the vertex of the affine cone over \( X \).

**Proof of Proposition 7.9.** Let \( \{D_+(f_\lambda)\} \) be a basic open affine cover of \( X \) which trivializes \( \mathcal{L} \). So each \( \mathcal{L}|_{D_+(f_\lambda)} \cong t_\lambda \mathcal{O}_{X|D_+(f_\lambda)} \), where \( t_\lambda \) is a local generator for \( \mathcal{L} \) over \( D_+(f_\lambda) \). Because \( \mathcal{L} = S(1) \), the local generator \( t_\lambda \) can be taken to be a degree one element of \( S[1/f_\lambda] \). Note that also \( t_n^{\lambda} \) is a local generator for \( \mathcal{L}^n \cong S(n) \) over \( D_+(f_\lambda) \) for all \( n \in \mathbb{Z} \), so that as an \( S[1/f_\lambda] \)-module, \( \mathcal{L}^n(D_+(f_\lambda)) \) is generated by \( t_n^{\lambda} \) for all \( n \in \mathbb{Z} \). Summarizing,

\[
S[1/f_\lambda] = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^n(D_+(f_\lambda)) = [S[1/f_\lambda]]_0 [t_\lambda, t^{-1}_\lambda] \cong \mathcal{O}_X(D_+(f_\lambda)) \otimes_k k[t_\lambda, t^{-1}_\lambda].
\]

This describes the affine chart \( D(f_\lambda) \) of \( \text{Spec} \ S \setminus \{m\} \) as isomorphic to the product of \( \text{Spec} k[t_\lambda, t^{-1}_\lambda] \) with the affine chart \( D_+(f_\lambda) \) of \( X \). Because together the \( D(f_\lambda) \) cover \( \text{Spec} \ S \setminus \{m\} \), the ring inclusions

\[
[S[1/f_\lambda]]_0 \hookrightarrow S[1/f_\lambda]
\]

induce a map

\[
\text{Spec} \ S \setminus \{m\} \to \text{Proj} \ S
\]

such that the preimage of \( D_+(f_\lambda) \) is isomorphic to \( D_+(f_\lambda) \times_k k^\times \). This shows that \( \text{Spec} \ S \setminus \{m\} \) is a \( k^\times \) bundle over \( X \), as claimed. \( \square \)

We are ready to prove the converse of Theorem 7.2.
Theorem 7.12. Let $X$ be a Frobenius split (respectively, globally $F$-regular) projective variety. Then for any ample invertible sheaf $\mathcal{L}$, the section ring $S(X, \mathcal{L})$ is Frobenius split (respectively, globally $F$-regular).

Proof. First assume that $X$ is Frobenius split. So there exists $\phi$ such that the composition

$$\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$$

is the identity. Now tensor with the sheaf of algebras $\bigoplus_{n \in \mathbb{N}} \mathcal{L}^n$ and use the projection formula to get

$$\bigoplus_{n \in \mathbb{N}} \mathcal{L}^n \xrightarrow{F} F_*(\bigoplus_{n \in \mathbb{N}} \mathcal{L}^n)^{pm} \xrightarrow{\phi} \bigoplus_{n \in \mathbb{N}} \mathcal{L}^n.$$ 

Taking global sections, we have the identity composition of $S$-modules

$$S \xrightarrow{F} F_*(S^{(p)}) \xrightarrow{\phi} S,$$

where $S^{(p)}$ denotes the $p^{th}$ Veronese subring of $S$. Since the Veronese subrings split off $S$, let $\pi \in \text{Hom}_{S^{(p)}}(S, S^{(p)})$ be a splitting of $S^{(p)} \hookrightarrow S$. Now define $\psi$ to be the composition

$$F_*S \xrightarrow{F_*\pi} F_*S^{(p)} \xrightarrow{\phi} S,$$

It is easy to check that $\psi$ is a splitting of the Frobenius map $S \to F_*S$. This completes the proof that if a projective variety $X$ is Frobenius split, so are all its section rings.

Now say that $X = \text{Proj} S$ is globally $F$-regular. Since $X$ is reduced, its regular locus is non-empty. So there exists a basic open affine $D_+(c) \subset X$ which is regular (where $c$ is some homogeneous element of $S$ of positive degree). This means that $[S^{[\frac{1}{c}]}]_0$ is regular, and so is the faithfully flat extension $S^{[\frac{1}{c}]}$ (Proposition 7.9). Thus $c$ can be used to test $F$-regularity for $S$ (Theorem 4.13).

Let $D$ be the effective divisor on $X$ defined by the homogeneous ideal $\langle c \rangle$ of $S$. In this case, the graded module corresponding to the coherent sheaf $\mathcal{O}_X(D)$ is precisely $\frac{1}{c}S$ (which is isomorphic as a graded $S$-module to the shifted module $S^{(\deg c)}$). By Lemma 7.4, the graded module corresponding to $F_*\mathcal{O}_X(D)$ is $[F_*^{\frac{1}{c}}S]_Z$.

Since $X$ is globally $F$-regular, there is an $e$ such that the natural map

$$\mathcal{O}_X \to F_*^{\frac{1}{c}}\mathcal{O}_X(D) \quad 1 \mapsto F_*^{\frac{e}{c}}1$$

splits. Tensoring with the sheaf of $\mathcal{O}_X$-algebras $\bigoplus \mathcal{L}^n$, we have that

$$\bigoplus_{n \in \mathbb{N}} \mathcal{L}^n \xrightarrow{\bigoplus_{n \in \mathbb{N}} F_*^{\frac{1}{c}}\mathcal{O}_X(D)} \bigoplus_{n \in \mathbb{N}} F_*^{\frac{1}{c}}\mathcal{O}_X(D) \quad 1 \mapsto F_*^{\frac{e}{c}}1,$$

splits as well. Taking global sections, we have that

$$S \to [F_*^{\frac{1}{c}}S]_Z \quad 1 \mapsto F_*^{\frac{e}{c}}1$$

splits, as well, where we’ve used Serre’s correspondence and Lemma 7.4 to identify the graded $S$-module module corresponding to $F_*^{\frac{1}{c}}\mathcal{O}_X(D)$; let $\phi \in \text{Hom}_S([F_*^{\frac{1}{c}}S]_Z, S)$ be a splitting. Also the natural inclusion $[F_*^{\frac{1}{c}}S]_Z \hookrightarrow F_*^{\frac{1}{c}}S$ splits (Remark 7.5);
let $\psi \in \text{Hom}_S(F^e_s(\frac{1}{c} S, F^e_s(\frac{1}{c} S))_\mathbb{Z})$ be a splitting. Then the map $S \longrightarrow F^e_s(\frac{1}{c} S)$ is split by the composition
$$F^e_s(\frac{1}{c} S) \overset{\psi}{\longrightarrow} [F^e_s(\frac{1}{c} S)]_\mathbb{Z} \overset{\phi}{\longrightarrow} S \quad F^e_s1 \mapsto F^e_s1 \mapsto 1.$$  
So the map
$$S \overset{F^e_s}{\longrightarrow} F^e_s S \quad 1 \mapsto F^e_s c$$
splits (Remark 6.20) as well. Finally, since $c$ is a test element for $S$, we can conclude that $S$ is F-regular (Theorem 4.13). 

There is a notion of “test elements” for global F-regularity:

**Theorem 7.13.** Let $X$ be a projective variety. Suppose that $X$ admits an ample effective divisor $D$ such that $X \setminus D$ is F-regular. Then if there exists $\phi \in \text{Hom}(F^e_s \mathcal{O}_X(D), \mathcal{O}_X)$ such that $\phi(F^e_s 1) = 1$, then $X$ is globally F-regular.

We leave the proof as an exercise; the point is that since $D$ is ample, we can use $D$ to build a section ring, and then the argument is similar to the proof of Theorem 7.12.
CHAPTER 4

Criteria for Frobenius Splitting

We have seen that Frobenius split schemes, and their strengthened brethren the F-regular schemes, have nice properties. Therefore, it behooves us to develop criteria for identifying Frobenius splitting. In this Chapter, we prove two such criteria.

The first is a criterion for Frobenius splitting first proved by Hochster and Roberts. We state a global form of their criterion that was exploited to great effect by Mehta and Ramanathan, who also provided the simple proof we give in Section 2. In Section 3, we develop Hochster and Robert’s local theory of F-purity, a generalization of Frobenius splitting that works well even for non-F-finite rings, and explain how to use it to deduce Hochster and Roberts original proof of Theorem 1.1.

In Section 4, we state a criterion due to Richard Fedder, a student of Hochster’s. Fedder’s criterion is an explicit criterion for local or global Frobenius splitting in terms of the explicit defining equations of a variety, and it has many immediate corollaries. Proving it will require a deep exploration of the structure of the module \( \text{Hom}_R(F_*R, R) \), which we undertake in Section 5.1.

Both criteria admit analogs to check for F-regularity which we discuss in the final section.

1. Ordinariness

In its simplest form, the criterion of Hochster and Roberts states that:

**Theorem 1.1.** Let \( X \) be a smooth projective variety of characteristic \( p \) and dimension \( d \). Then \( X \) is Frobenius split if and only if the natural map induced by Frobenius

\[
H^d(X, \omega_X) \rightarrow H^d(X, \omega_X \otimes F_* \mathcal{O}_X)
\]

is injective. Put differently, \( X \) is Frobenius split if and only if the natural map

\[
H^d(X, \omega_X) \rightarrow H^d(X, \omega_X^p)
\]

is injective.

**Remark 1.2.** There is a canonical sheaf \( \omega_X \) defined on every projective variety \( X \), and Theorem 1.1 (as stated in the first sentence) actually holds in complete generality. The only difficulty is to say what we mean by \( \omega_X \). Of course, for smooth \( X \), we define \( \omega_X = \bigwedge^{\dim X} \Omega_{X/k} \) where \( \Omega_{X/k} \) is the sheaf of Kähler differentials. Theorem 1.1 is interesting enough in the smooth case, so the reader is encouraged to focus on that

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1Fedder was one of Hochster’s first PhD students at UM (Mel had a few at Minnesota before moving to Michigan). This criterion was in his 1981 PhD thesis, and he is greatly amused that it has gained so much traction. Fedder worked as a mathematician for a few years at the University of Missouri before deciding to go back to school for a JD. He now has his own law firm in Carbondale, IL.
case if $\omega_X$ is not familiar in these more general settings. Still, it is fairly easy to say what $\omega_X$ is in different settings; for example, for normal $X$, we can define $\omega_X$ to be the unique reflexive extension of $\omega_U$ on the smooth set $U$. See §2.1.

**Remark 1.3.** By Serre duality, $H^d(X, \omega_X)$ is dual to $H^0(X, \mathcal{O}_X)$, which is isomorphic to $k$ (assuming $X$ is connected). So saying that the map in Theorem 1.1 is injective is the same as saying it is non-zero.

**Example 1.4.** A nice application of Theorem 1.1 is that an elliptic curve is Frobenius split if and only if Frobenius acts injectively on $H^1(X, \mathcal{O}_X)$, that is, if and only if $X$ is ordinary. Indeed, for an elliptic curve, $\omega_X = \mathcal{O}_X$, so this statement is an immediate corollary of Theorem 1.1.

This completes our classification of smooth projective curves: a curve is Frobenius split if and only if it is genus zero or an ordinary elliptic curve. [We had already completed our classification of globally F-regular curves: a smooth projective curve is F-regular if and only if its genus is zero.]

**Example 1.5.** Generalizing the previous application, assume that $X \subset \mathbb{P}^n$ is a hypersurface of degree $n+1$. Then $\mathcal{O}_X \cong \omega_X$, and $X$ is Frobenius split if and only if the Frobenius map acts injectively on the one dimensional vector space $H^{\dim X}(X, \mathcal{O}_X)$. [This works even if $X$ is singular.]

**Example 1.6.** An abelian variety of characteristic $p$ is Frobenius split if and only if the Frobenius action $H^{\dim X}(X, \mathcal{O}_X) \xrightarrow{F} H^{\dim X}(X, \mathcal{O}_X)$ is injective (equivalently, non-zero). Again, the point is that $\omega_X = \mathcal{O}_X$ for an abelian variety.

**Remark 1.7.** Hochster and Roberts actually proved a local form of Theorem 1.1, namely Theorem 3.10, from which Theorem 1.1 follows by considering the local ring at the vertex of the cone over $X$ with respect to some section ring. However, it is also possible to give a direct proof of Theorem 1.1 for projective varieties, which was observed by Mehta and Ramanathan a decade later. We will first explain the direct proof in for projective varieties. To state the Hochster and Huneke’s local local version, we will need to review purity of maps and injective modules.

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2Even for a non-Cohen-Macaulay variety, Serre duality always holds “at the top spot.”
2. The Proof of Theorem 1.1

In this section, we prove Theorem 1.1 using mostly Serre duality. For readers more inclined towards commutative algebra, we give an alternate (more general) proof in the next section.

Fix a smooth projective variety $X$ of prime characteristic $p$. To split the map $\mathcal{O}_X \to F_*\mathcal{O}_X$ is equivalent to split the dual map $(F_*\mathcal{O}_X)^\vee \to \mathcal{O}_X^\vee \cong \mathcal{O}_X$ where the dual here is the sheaf dual $\text{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$. For this, we want to find a map of sheaves $\mathcal{O}_X^\vee \to (F_*\mathcal{O}_X)^\vee$ such that composing with the map $(F_*\mathcal{O}_X)^\vee \to \mathcal{O}_X^\vee$ we have the identity map of $\mathcal{O}_X^\vee$.

A map of sheaves $\mathcal{O}_X^\vee \cong \mathcal{O}_X \to (F_*\mathcal{O}_X)^\vee$ is a global section of $H^0(X, (F_*\mathcal{O}_X)^\vee)$. Such a map is a splitting if and only if the induced composition map in $\mathcal{O}_X^\vee \to \mathcal{O}_X^\vee$ is the identity map, which is to say, given by the global section $1 \in H^0(X, \mathcal{O}_X^\vee)$. Thus the variety $X$ is Frobenius split if and only if the induced map on global sections $H^0(X, (F_*\mathcal{O}_X)^\vee) \to H^0(X, \mathcal{O}_X^\vee)$ is surjective. By Serre duality, this is equivalent to $H^d(X, \omega_X) \to H^d(X, F_*\mathcal{O}_X \otimes \omega_X)$ being injective. This completes the proof of Theorem 1.1.

**Remark 2.1.** Theorem 1.1 holds for every arbitrary projective variety—we do not need to assume $X$ is smooth. Indeed, all we used in the proof above is that there is a “canonical module” $\omega_X$ for which Serre duality holds “at the top spot.” As it turns out, this is the case for every projective variety. For a Cohen-Macaulay $X$, Serre duality is discussed in Hartshorne, and a construction of $\omega_X$ is discussed there. But even that is more than we need: we only need the duality between $H^0$ and $H^{\dim X}$.

To explain why Serre duality also holds in the non-Cohen-Macaulay case “at the top spot” requires a digression. There are two possible ways to approach it; we only sketch these.

2.0.1. *The canonical module using Grothendieck duality.* For readers familiar with Grothendieck duality, we sketch as follows. Every variety has a dualizing complex. By definition, this is a bounded complex $\omega_X^\bullet$ of coherent $\mathcal{O}_X$-modules which is quasi-isomorphic to a bounded complex of injective sheaves and such that the natural map $\mathcal{O}_X \to R\text{Hom}_R(\omega_X^\bullet, \omega_X^\bullet)$ is an isomorphism. This complex is concentrated in one homological spot if (and only if) $X$ is Cohen-Macaulay; in this case, the one non-zero module in the dualizing complex is the dualizing module. It agrees with the usual sheaf of top Kähler differentials on the smooth locus of $X$.
In the non-Cohen-Macaulay case, the dualizing complex has a final non-zero module $\omega_X$. If we compare dualizing using the entire complex to dualizing with just the last module $\omega_X$, we see that we get the same result in the last spot. This is Serre duality in the last spot, as above.

2.0.2. The canonical module using Matlis duality. Alternatively, we can take a more commutative algebraic approach, and use Matlis duality. Here, we first replace $X$ by some section ring $S = S(X, \mathcal{L})$ with respect to some ample $\mathcal{L}$ (or its localization at the unique homogeneous maximal ideal $m$). Matlis duality means $\text{Hom}_S(-, E)$ where $E$ is an injective hull of the residue field $S/m$. Then the canonical module of $S$ is defined as any $S$ module $\omega_S$ whose Matlis dual is isomorphic to the local cohomology module $H^\dim S_m(S)$. Such an $\omega_S$ may not exist for an arbitrary local ring, but they do exist for (localizations of) finitely generated algebras over a field, such as our section ring $S$. In the section ring case, both $\omega_S$ and $H^\dim S_m(S)$ are graded and Matlis duality gives a (graded) perfect pairing

$$\omega_S \times H^\dim S_m(S) \rightarrow E.$$

Now, we can define $\omega_X$ to be the sheaf on $X$ corresponding to the graded $S$-module $\omega_S$. Restricting the pairing above to degrees $(n, -n)$ produces a pairing

$$H^0(X, \omega_X \otimes \mathcal{L}^n) \times H^\dim X (X, \mathcal{L}^{-n}) \rightarrow [E]_0 = H^\dim X (X, \omega_X) \cong S/m.$$

We flesh some of this out in the next section.

3. The Hochster-Robert’s Criterion for purity

In this section we develop the Hochster-Roberts Criterion for Frobenius splitting in the local case, which is more commutative algebro-geometric approach. We also present Hochster and Robert’s generalization of the notion of Frobenius splitting to F-purity, which is a closely related notion that is better behaved for non-F-finite rings. We will then sketch how to use the local version of the criterion to deduce the global one. Readers mostly interested in case of smooth projective varieties, or who have little training in commutative algebra, can skip this section.

**Definition 3.1.** A map of rings $R \rightarrow S$ is **pure** if for every $R$-module $M$, the induced map $M \rightarrow S \otimes_R M$ is injective.

**Definition 3.2.** A commutative ring $R$ of characteristic $p$ is said to be **F-pure** if the Frobenius map $R \rightarrow F_* R$ is pure.

Clearly, if a ring map $R \rightarrow S$ splits, then it is pure. Thus F-purity can be viewed as a weakening Frobenius splitting. On the other hand, we will prove that F-purity and Frobenius splitting are equivalent for F-finite rings (Corollary 3.9).

Hochster and Roberts proved the following simple criterion for purity, showing we only need to check injectivity is preserved for one module:
**Theorem 3.3.** Let \((R,m)\) be a Noetherian local ring. Let \(E\) denote the injective hull of the residue field \(R/m\). A ring map \(R \rightarrow S\) is pure if and only if the induced map
\[ E \rightarrow S \otimes_R E \]
is injective.

To understand and prove Theorem 3.3, we need to review injective modules.

### 3.0.1. Injective Modules

We recall some basic facts about injective modules. All details can be found in Hochster’s Notes on Local Cohomology.

**Definition 3.4.** An \(R\)-module \(I\) is injective if \(\text{Hom}_R(\cdot, I)\) is an exact functor of \(R\)-modules.

Since such Hom functors are always left exact, we can define an injective module \(I\) as one so that \(\text{Hom}_R(\cdot, I)\) is right exact.

**Exercise 3.1.** Every injective map of \(R\)-modules from an injective module splits. This is dual to the corresponding statement for projective modules: Every surjective map of \(R\)-modules to a projective module splits.

**Definition 3.5.** Let \(R\) be any ring and \(M\) any \(R\)-module. Then \(M\) admits an **injective hull**, that is, an injective module \(E_R(M)\) containing \(M\) as a submodule with the following equivalent properties:

(a) If \(E\) is some other injective \(R\) module containing \(M\), then \(E\) contains \(E_R(M)\) (in which case \(E_R(M) \hookrightarrow E\) splits);

(b) Every non-zero element \(\eta \in E_R(M)\) has a non-zero \(R\)-multiple in \(M\) (that is, \(M \hookrightarrow E_R(M)\) is an **essential extension**).

Over a commutative Noetherian ring \(R\), the injective modules are well understood: every injective module is direct sum of injective hulls \(E_R(R/P)\) where \(P \in \text{Spec } R\). Indeed, the isomorphism types of **irreducible injective modules** (those that can not be decomposed into a direct sum of two proper submodules) are precisely those of the form \(E_R(R/P)\) for \(P \in \text{Spec } R\).

**Example 3.6.** Over the ring \(\mathbb{Z}\), the module \(\mathbb{Q}\) is the injective hull of \(\mathbb{Z}\). The module \(\mathbb{Q}/\mathbb{Z}\) is also injective, and is the direct sum of the injective hulls of \(\mathbb{Z}/p\mathbb{Z}\) for all (non-zero) primes \(p \in \mathbb{Z}\). The injective hull \(E_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z})\) can be identified with the injective hull of the residue field of the local ring \(\mathbb{Z}_{(p)}\). It is the cokernel of the \(\mathbb{Z}\)-module map
\[ \mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}. \]
3.0.2. The injective hull of the residue field. The injective hull of the residue field of a Noetherian local ring \((R, m)\) is of particular importance. The following properties follow from the definition:

(a) Every non-zero \(\eta \in E_R(R/m)\) has a non-zero \(R\)-multiple in \(R/m\) (since \(R/m \subset E_R(R/m)\) is an essential extension);

(b) The only associated prime of \(E_R(R/m)\) is \(m\). Indeed, if there is an injection \(R/P \hookrightarrow E_R(R/m)\), then the image of 1, and every non-zero multiple of it, has annihilator \(P\). So by (a), \(P = m\).

(c) Every element of \(E_R(R/m)\) is killed by a power of \(m\) (since by (a), \(\eta \in E_R(R/m)\) has non-zero annihilator, and by (b) that annihilator is \(m\)-primary).

Remark 3.7. We will give an explicit description of \(E_R(R/m)\) in the case where \(R\) is the localization of a section ring of projective variety at its unique homogeneous maximal ideal. See Section ??.

Proof of Theorem 3.3. Say that \(R \to S\) is not pure. Then there exists some \(R\)-module \(M\) such that the induced map \(M \to S \otimes_R M\) is not injective. By viewing \(M\) as the direct limit of its finitely generated submodules, without loss of generality, the module \(M\) may be assumed finitely generated. Take any non-zero \(x \in M\) in the kernel. Since \(\bigcap_{t \in \mathbb{N}} m^t M = 0\), there exists \(t\) such that \(x \notin m^t M\).

Consider the image \(\bar{x}\) of \(x\) in \(M/m^t M\). We know that \(\bar{x}\) is not zero, but that \(\bar{x}\) is in the kernel of the map

\[
M/m^t M \to S \otimes_R M/m^t M = S \otimes_R M/(\text{im} S \otimes_R m^t M).
\]

Thus if \(R \to S\) is not pure, we may assume without loss of generality that there is a finite length \(R\)-module \(M\) for which the map \(M \to S \otimes_R M\) is not injective.

Let \(E\) be the injective hull of \(M\). Because \(M\) has finite length, \(E\) is the direct sum of copies of \(E_R(R/m)\). Indeed, if some other \(E_R(R/P)\) appears in \(E\), then there are injections \(R/P \hookrightarrow E_R(R/P) \hookrightarrow E\). If we let \(\eta\) be the image of 1 \(\in R/P\) in \(E\), we see that the annihilator of \(\eta\) and all its non-zero multiples is \(P\). But \(\eta\) has a non-zero multiple in \(M\), where every non-zero element has annihilator \(m\). Thus \(P = m\).

Consider the commutative diagram

\[
\begin{array}{ccc}
M & \hookrightarrow & E(M) \\
\downarrow & & \downarrow \\
M \otimes_R S & \to & E(M) \otimes_R S \\
\end{array}
\]

where the top horizontal arrow is the inclusion of \(M\) in its injective hull, and the vertical arrows are the naturally induced maps by tensoring with \(R \to S\). Because \(E(M) \cong \bigoplus E_R(R/m)\), the right downward arrow is injective by our hypotheses. The
commutativity of the diagram implies that also the left downward arrow is injective. This contradiction finishes the proof of Theorem 3.3. □

3.0.3. The equivalence of purity and splitting for finite maps. An important fact is that F-purity and Frobenius splitting are the same in the F-finite case:

**Corollary 3.8.** A Noetherian F-finite ring $R$ (of prime characteristic) is F-pure if and only if it is Frobenius split.

This follows from the following general fact, due to Hochster and Roberts:

**Theorem 3.9.** [?, Corollary 5.3] Let $R$ be a Noetherian ring. Then a finite extension $R \hookrightarrow S$ is pure if and only if it is split in the category of $R$-modules.

As an immediate corollary, we have a local criterion for Frobenius splitting.

**Corollary 3.10.** Let $(R, m)$ be a Noetherian local F-finite ring of characteristic $p$. Let $E$ denote the injective hull of the residue field. Then $R$ is Frobenius split if and only if the Frobenius map $R \to F_\ast R$ remains injective after tensoring with $E$.

The proof of Theorem 3.9 makes use of the following homological lemma:

**Lemma 3.11.** Let $R$ be any ring. Suppose that $M$ is a finitely presented $R$-module, ie, that we have an exact sequence

\[ R^n \xrightarrow{\alpha} R^m \longrightarrow M \longrightarrow 0. \]

Define $M'$ to be the cokernel of the transpose of $\alpha$, that is, $M'$ is induced by the dual sequence

\[ \text{Hom}_R(R^m, R) \xrightarrow{\alpha^t} \text{Hom}_R(R^n, R) \longrightarrow M' \longrightarrow 0. \]

Then for any short exact sequence of $R$-modules

\[ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0, \]

we have an isomorphism

\[ \text{coker}[\text{Hom}_R(M, B) \to \text{Hom}_R(M, C)] \cong \ker[M' \otimes_R A \to M' \otimes B]. \]

**Proof of Lemma 3.11.** Using the fact that $\text{Hom}_R(R^d, N) \cong \text{Hom}_R(R^d, R) \otimes_R N$ for any $R$-module $N$ and any $d \in \mathbb{N}$, we have the following commutative diagram
with exact columns and (mostly) exact rows:

\[ \begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\Hom_R(M, A) & \Hom_R(M, B) & \Hom_R(M, C) \\
\downarrow & \downarrow & \downarrow \\
\Hom_R(R^n, R) \otimes_R A & \Hom_R(R^n, R) \otimes_R B & \Hom_R(R^n, R) \otimes_R C \\
\downarrow \alpha^r \otimes \text{id}_A & \downarrow \alpha^r \otimes \text{id}_B & \downarrow \alpha^r \otimes \text{id}_C \\
\Hom_R(R^n, R) \otimes_R A & \Hom_R(R^n, R) \otimes_R B & \Hom_R(R^n, R) \otimes_R C \\
\downarrow & \downarrow & \downarrow \\
M' \otimes_R A & M' \otimes_R B & M' \otimes_R C \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array} \]

We can now show using a Snake Lemma like diagram chase\(^3\) (and the fact that \(\beta\) is injective and \(\gamma\) is surjective) that there is a map \(d\) such that the sequence

\[ \Hom_R(M, A) \to \Hom_R(M, B) \to \Hom_R(M, C) \xrightarrow{d} M' \otimes_R A \to M' \otimes B \to M' \otimes C \]

is exact. The statement follows immediately. \(\square\)

**Proof of Theorem 3.9.** First we make a general conclusion from Lemma 3.11. A finitely generated module over a Noetherian ring \(R\) can always be viewed as the cokernel of some matrix \(\alpha\), or as the cokernel of the transpose of some matrix \(\alpha\), as in the lemma. Therefore, demanding injectivity of \(M' \otimes A \to M' \otimes B\) as we range over all finitely generated modules \(M'\) is equivalent to demanding surjectivity for all \(\Hom_R(M, B) \to \Hom_R(M, C)\) for all finitely generated \(R\)-modules \(M\) (where \(C\) is the cokernel of \(A \hookrightarrow B\), as Lemma 3.11).

Consider the short exact sequence of finitely generated \(R\)-modules

\[ 0 \to R \to S \to S/R \to 0. \]

Given that \(R \to S\) is pure, we know that \(\ker[M' \otimes_R R \to M' \otimes S] = 0\) for all finitely generated \(M'\). According to the lemma, this means that \(\coker[\Hom_R(M, S) \to \Hom_R(M, S/R)] = 0\) for all finitely generated \(M\). In particular, take \(M = S/R\). We have that

\[ 0 \to \Hom_R(S/R, R) \to \Hom_R(S/R, S) \to \Hom_R(S/R, S/R) \to 0 \]

\(^3\)or see [?], Chap. I, §1 Prop 2].
is exact. This means that the identity map of $\text{Hom}_R(S/R, S/R)$ lifts to some $\phi \in \text{Hom}_R(S/R, S)$. This $\phi : S/R \to R$ is a splitting of the natural surjection $S \to S/R$. This means that the exact sequence
\[ 0 \to R \to S \to S/R \to 0. \]
is split. \hfill \Box

3.0.4. *The Application to projective Varieties.* Finally, to deduce the Hochster-Roberts criterion for an Frobenius split projective variety, we make use of our results relating the global properties of projective varieties to local properties “at the vertex of the cone” defined by a section ring.

**Notation 3.12.** Fix a smooth projective variety over an F-finite field $k$. Choose any ample invertible sheaf $\mathcal{L}$ on $X$ and let $S$ denote the corresponding section ring $S(X, \mathcal{L})$. Let $m$ be its unique homogeneous maximal ideal.

**Definition 3.13.** The **graded canonical module** for the section ring $S$ is the $\mathbb{Z}$-graded $S$-module
\[ \omega_S := \bigoplus_{n \in \mathbb{Z}} H^0(X, \omega_X \otimes \mathcal{L}^n). \]
Equivalently, $\omega_S$ is the unique saturated graded $S$-module corresponding to the coherent sheaf $\omega_X$ under Serre’s correspondence.

**Remark 3.14.** Definition 3.13 is not the usual definition, which we avoid here because not every student has taken a course on local cohomology. However, for students who have seen some local cohomology, it is not a difficult exercise to verify that Definition 3.13 agrees with any other definition (or see [?], p5). [By definition, a canonical module is one whose Matlis dual is $H_{m}^{\dim R}(R)$, but this begs further questions: what is local cohomology? what is Matlis dual? All this is covered in standard UM courses on local cohomology, and is written in detail in Hochster’s Notes on Local Cohomology].

**Black Box Fact.** Let $S = S(X, \mathcal{L})$ be a section ring for a projective variety $X$. The injective hull $E_S(S/m)$ of the residue field $S/m$ for the local ring $S_m$ is isomorphic, as a $\mathbb{Z}$-graded $S$-module, to
\[ H_{m}^{\dim S}(\omega_S) \cong \bigoplus_{n \in \mathbb{Z}} H^{\dim X}(X, \omega_X \otimes \mathcal{L}^n). \]
Here the residue field $S/m$ embeds as the degree zero part of either of these modules, which are isomorphic as graded $S$-modules.

**Remarks on Black Box.** For students who have studied local cohomology (eg, Mel’s notes), the first part of the blackbox should be recognized as a basic fact for

\[ \text{meaning satisfying Serre’s } S_2 \text{ condition.} \]
any local Noetherian ring \((R, m)\): if \(R\) has a canonical module \(\omega_R\), then \(H^{\dim R}_m(\omega_R)\) is an injective hull for the residue field \(E_R(R/m)\). The relationship between local cohomology and sheaf cohomology is explained below.

3.0.5. Local Cohomology and Sheaf Cohomology. Serre’s correspondence between graded modules over a section ring \(S\) and quasicoherent sheaves on \(\text{Proj} \ S\) extends to a nice correspondence between local cohomology modules at the vertex \(m\) of the cone \(\text{Spec} \ S\) and sheaf cohomology modules.

Let \(X\) be a projective variety with section ring \(S = S(X, \mathcal{L})\) with respect to a fixed ample invertible sheaf \(L\). Let \(F\) be a coherent \(\mathcal{O}_X\)-module, and let \(M = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{L}^n)\) be the corresponding graded \(S\)-module.

**Proposition 3.15.** For all \(i \geq 2\), the local cohomology module

\[ H^i_m(M) \]

is a \(\mathbb{Z}\)-graded \(S\) module. Its graded pieces are precisely \(H^{i-1}(X, \mathcal{F} \otimes \mathcal{L}^n)\). That is, there is a natural degree-preserving isomorphism

\[ H^i_m(M) \cong \bigoplus_{n \in \mathbb{Z}} H^{i-1}(X, \mathcal{F} \otimes \mathcal{L}^n) \]

of graded \(S\)-module.

**Proof.** This is easy to see if one knows how to compute local cohomology.

For any module \(M\) over a local Noetherian ring \((R, m)\), the local cohomology modules \(H^i_m(M)\) are the derived functors of the “global sections with support” functors \(H^0_m(\mathcal{F})\) defined by

\[ H^0_m(M) := \{ m \in M \mid m \text{ is supported at } m \} \]

The local cohomology modules describe the obstruction to extending sections of the sheaf \(\widetilde{M}\) on the affine scheme Spec \(R\) from the open set \(U = \text{Spec} \ R \setminus \{m\}\) to Spec \(R\). That is, we have an exact sequence

\[ 0 \to H^0_m(M) \to H^0(\text{Spec} \ R, \widetilde{M}) \xrightarrow{\text{restrict}} H^0(U, \widetilde{M}) \to H^1_m(M) \to H^1(\text{Spec} \ R, \widetilde{M}) \to \]

\[ H^1(U, \widetilde{M}) \to H^2_m(M) \to H^2(\text{Spec} \ R, \widetilde{M}) \to H^2(U, \widetilde{M}) \to \ldots \]

This long exact sequence shows that there are natural isomorphisms

\[ H^i(U, M) \to H^{i+1}_m(M) \]

for all \(i \geq 1\), because the higher cohomologies of quasi-coherent sheaves vanish on an affine scheme.

Returning to our section ring \((S, m)\), let \(M\) be a graded \(S\)-module corresponding to the sheaf \(\mathcal{F}\). Consider a homogeneous system of parameters \(f_0, \ldots, f_d\). The open sets \(D(f_0), \ldots, D(f_d)\) form an open affine cover for \(U = \text{Spec} \ S \setminus \{m\}\), and the open
sets \( D_+(f_0), \ldots, D_+(f_d) \) form an open affine cover for \( X \). Writing down the Čech complex that computes the cohomology \( H^i(\text{Spec } S \setminus \{m\}, \widetilde{M}) \) from the cover \( \{D(f_i)\} \) we observe that its zero graded piece is precisely the Čech complex from the cover \( \{D_+(f_i)\} \) of \( \text{Proj } S \) computing \( H^{i-1}(X, F) \). Similarly, its \( n \)-graded piece is the Čech complex that computes \( H^{i-1}(X, F \otimes \mathcal{L}^n) \). This complete the proof. \( \square \)

We are ready to prove Theorem \[1.1\]. We first need a useful lemma.

**Lemma 3.16.** If \( S \) is a finitely generated \( \mathbb{N} \)-graded algebra over an \( F \)-finite field \( k \), then \( S \) is Frobenius split if and only if \( S_m \) is Frobenius split, where \( m \) is the unique homogeneous maximal ideal of \( S \).

**Proof.** We sketch an argument in the case where \( k \) is infinite.\[5\] The reduction to that case is left as an exercise. [Hint: tensor over \( k \) with the infinite field \( k(t) \).]

Every \( \mathbb{N} \)-graded ring \( S \) over \( k \) has a natural action by \( k^\times \): we define the action of \( \lambda \in k^\times \) on an element \( s \in S_d \) by \( \lambda \cdot s = \lambda^d s \). It is easy to check that this defines a group action on the ring \( S \), and that an ideal \( I \subset S \) is invariant under this action if and only if \( I \) is homogeneous (this is where we use that \( k \) is infinite).

Since \( k^\times \) acts by ring automorphisms on \( S \), there is an induced action on \( \text{Spec } S \), and if some \( \lambda \in k^\times \) sends \( P \in \text{Spec } S \) to \( Q \in \text{Spec } S \), then \( S_P \cong S_Q \). This means that the (open) locus of Frobenius split points in \( \text{Spec } S \) is invariant under the \( k^\times \)-action, as is the (closed) locus of non-Frobenius split points of \( \text{Spec } S \). So without loss of generality, we can assume that the defining ideal \( I \subset S \) of the non-Frobenius split locus is homogeneous. Thus, if the non-Frobenius split locus is non-empty, it must include the unique homogeneous maximal ideal \( m \). That is, \( S \) is Frobenius split at every point if and only if it is Frobenius split at the unique homogeneous maximal ideal. \( \square \)

**Proof of Theorem \[1.1\]** Fix a section ring \( S = S(X, \mathcal{L}) \) for \( X \). Let \( m \) denote its unique homogeneous maximal ideal.

The variety \( X \) is Frobenius split if and only if the cone \( \text{Spec } S \) is Frobenius split, which holds if and only if \( S_m \) if Frobenius split.

The Frobenius splitting of \( S_m \) is equivalent to the injectivity of the map \( E \to E \otimes_S F_*S \), where

\[
E = E_S(S/m) = H^\dim S_m(\omega_S) = \bigoplus_{n \in \mathbb{Z}} H^\dim X(X, \omega_X \otimes \mathcal{L}^n)
\]

\[5\]This same argument works for any open ring property, as you will see from the proof.
is an injective hull of the residue field $S/m$. Observe that $E$ is zero in positive degrees, and that in degree zero, $E$ is precisely $S/m \cong H^{\dim X}(X, \omega_X)$. Thus, the extension

$$S/m \cong H^{\dim X}(X, \omega_X) \hookrightarrow \bigoplus_{n \in \mathbb{Z}} H^{\dim X}(X, \omega_X \otimes \mathcal{L}^n) = E$$

is an essential extension.

Thus if some non-zero $\eta \in E$ is in the kernel of $E \to E \otimes_S F_*S$, then $\eta$ must have a non-zero multiple in $H^{\dim X}(X, \omega_X)$. So $E \to E \otimes_S F_*S$ is injective if and only if

$$H^{\dim X}(X, \omega_X) \hookrightarrow H^{\dim X}(X, F_*\mathcal{O}_X \otimes \omega_X)$$

is injective.

\[\square\]

**Remark 3.17.** Nothing in this proof uses the hypothesis that $X$ is smooth: if we adopt the commutative algebra definition of $\omega_S$ (as the graded $S$-module whose Matlis dual is $H^{\dim S}_M(S)$), then we can define $\omega_X$ to be $\widetilde{\omega}_S$ and the proof is precisely as above. The sheaf $\omega_X$ produced this way agrees with $\Lambda^d \Omega_{X/k}$ on the smooth locus (see [?].)

### 4. Fedder’s Criterion

Another powerful test for Frobenius splitting is due to Richard Fedder:

**Theorem 4.1.** Let $R = S/I$ be a quotient of a regular local $F$-finite ring $(S, m)$. Then $R$ is Frobenius split if and only if $(I^p : I) \not\subset m^p$. Here, the notation $(I^p : I)$ denotes the ideal of elements in $S$ which multiply $I$ into $I^p$.

Fedder’s Criterion is especially easy to use for hypersurfaces:

**Corollary 4.2.** Let $R = S/(f)$ be a hypersurface, where $(S, m)$ is an $F$-finite regular local ring and $f \neq 0$. Then $R$ is Frobenius split if and only if

$$f^{p-1} \notin m^p.$$

**Proof.** If $I \subset S$ is a principal ideal, then the mysterious term $(I^p : I)$ is easy to understand. If $I = \langle f \rangle$, then $(I^p : I) = \langle f^{p-1} \rangle$, and the corollary follows. \[\square\]

There is an analogous statement for complete intersections, whose proof we leave as an exercise:

**Exercise 4.1.** Let $R = S/(f_1, \ldots, f_r)$ be a complete intersection\footnote{Complete intersection means that $\dim R = \dim S - r$, or equivalently, that $f_1, \ldots, f_r$ form a regular sequence.} where $(S, m)$ is an $F$-finite regular local ring. Then $R$ is Frobenius split if and only if

$$(f_1 \cdots f_r)^{p-1} \notin m^p.$$
Fedder’s Criterion has many immediate applications:

**Example 4.3.** Consider the simple normal crossing divisor $D$ in $\mathbb{A}^n_K$ defined by $x_1x_2 \cdots x_n$ (where $K$ is F-finite). We claim that $D$ is Frobenius split at every point. Indeed, the coordinate ring of $D$ is $K[x_1, \ldots, x_n]/I$ where $I = \langle x_1x_2 \cdots x_n \rangle$. So $(I[p] : I) = \langle (x_1x_2 \cdots x_n)^{p-1} \rangle$. Now, since $(x_1x_2 \cdots x_n)^{p-1} \notin \langle x_1^p, \ldots, x_n^p \rangle$, Fedder’s Criterion tells us that $K[x_1, \ldots, x_n]/I$ and hence $D$ are Frobenius split at the origin. Similarly, since $(x_1x_2 \cdots x_n)^{p-1} \notin q^{[p]}$ for any maximal ideal $q$, we see that $D$ is Frobenius split at every point.

More generally, any reduced simple normal crossing divisor $D$ on a smooth variety $X$ is locally Frobenius split. The point is that at a point $q \in X$, the ring $\mathcal{O}_{D,q} \cong \mathcal{O}_{X,q}/\langle x_1x_2 \cdots x_r \rangle$, where $x_1, \ldots, x_r$ are part of a minimal set of generators for the maximal ideal $m$ of $\mathcal{O}_{X,q}$. It is not hard to see that $(x_1x_2 \cdots x_r)^{p-1} \notin m^{[p]}$, either by using the fact that the $x_i$ form a regular sequence or by completing at $m$ and arguing as in the case of $\mathbb{A}^n$ above.

**Example 4.4.** Consider an arbitrary graded hypersurface

$$\text{Spec } K[x_0, \ldots, x_n]/(f)$$

over an F-finite field of characteristic $p$. Fedder’s Criterion says that Spec $R$ is Frobenius split (at the origin, and hence everywhere, by Proposition 3.16) if and only if $f^{p-1} \notin \langle x_1^p, \ldots, x_n^p \rangle$. The same is true for the hypersurface in $\mathbb{P}^n$ defined by $f$ by Theorem 7.7.

Suppose that $f$ is homogeneous of degree at least $n + 2$. In this case, $\deg f^{p-1} \geq (p-1)(n+2)$. Considering a monomial of degree $(p-1)(n+2)$, the pigeon hole principal implies that at least one variable appears with exponent at least $p$. Thus $f^{p-1} \notin \langle x_0^p, x_1, \ldots, x_n^p \rangle$ and neither Spec $R$ nor Proj $R$ is Frobenius split, by Fedder’s Criterion. This recovers our earlier observation that a hypersurface of degree $d > n+1$ in $\mathbb{P}^n$ is never Frobenius split, which was deduced in Example 3.3 from Vanishing Theorems.

Now suppose $f$ has degree $n + 1$ in $n + 1$ variables. The pigeon hole principle implies that $f^{p-1}$, which is a sum of monomials of degree $(n+1)(p-1)$, has all terms already in $\langle x_0^p, x_1, \ldots, x_n^p \rangle$ with the possible exception of the term $(x_0x_1 \cdots x_n)^{p-1}$, which may or may not have non-zero coefficient. Thus $f^{p-1} \notin m^{[p]}$ if and only if the term $(x_0x_1 \cdots x_n)^{p-1}$ appears in $f^{p-1}$ with non-zero coefficient. That is, the hypersurface (in both $\mathbb{A}^{n+1}$ and $\mathbb{P}^n$) defined by $f$ is Frobenius split if and only if the term $(x_0x_1 \cdots x_n)^{p-1}$ appears in $f^{p-1}$ with non-zero coefficient.

**Example 4.5.** Now consider the cone over a planar elliptic curve, say, the affine hypersurface in $\mathbb{A}^3_k$ defined by a homogeneous polynomial $f$ of degree three. In this case, $f^{p-1}$ has degree $3p-3$. Note that every monomial of degree $3p-3$ is divisible by either $x^p$, $y^p$ or $z^p$, except for $(xyz)^{p-1}$. Thus $f^{p-1} \notin \langle x^p, y^p, z^p \rangle$ if and only the term
Let us examine more closely a particular elliptic curve, say with equation 

\[ f = x^3 + y^3 + z^3. \]

Using the trinomial expansion, we compute

\[ f^{p-1} = \sum_{i+j+k=p-1} \binom{p-1}{i,j,k} x^{3i}y^{3j}z^{3k}. \]

We see that if the term \((xyz)^{p-1}\) is to appear with non-zero coefficient, then we must have \(3i = 3j = 3k = p - 1\). That is, \(p \equiv 1 \pmod{3}\), and \(i = j = k = \frac{p-1}{3}\). In this case, the multinomial coefficient is

\[ \binom{p-1}{\frac{p-1}{3}, \frac{p-1}{3}, \frac{p-1}{3}} = \frac{(p-1)!}{\left(\frac{p-1}{3}\right)! \left(\frac{p-1}{3}\right)! \left(\frac{p-1}{3}\right)!}, \]

which clearly is not divisible by \(p\). Hence if \(p \equiv 1 \pmod{3}\), then the cone \(\text{Spec } k[x, y, z]/(x^3 + y^3 + z^3)\) is Frobenius split. On the other hand, if \(p \equiv 2 \pmod{3}\), the term \((xyz)^{p-1}\) does not appear, so \(\text{Spec } k[x, y, z]/(x^3 + y^3 + z^3)\) is not Frobenius split. Put differently, the elliptic curve in \(\mathbb{P}^2\) defined by \(x^3 + y^3 + z^3\) is ordinary when \(p \equiv 1 \pmod{3}\) and supersingular when \(p \equiv 2 \pmod{3}\).

\[ 5. \text{ The Proof of Fedder’s Criterion} \]

5.1. The \(F_* R\)-module structure of \(\text{Hom}_R(F^e_* R, R)\). We have seen that the \(R\)-module \(\text{Hom}_R(F^e_* R, R)\) is important to understanding Frobenius splitting. To look deeper, we must also understand the structure of \(\text{Hom}_R(F^e_* R, R)\) as a module over the ring \(F^e_* R\).

In general, if \(S\) is any algebra over any ring \(R\) and \(M\) is any \(R\)-module, the \(R\)-module \(\text{Hom}_R(S, M)\) has a (right) \(S\)-module structure that comes from the action of \(S\) on the source. That is, \(s \in S\) acts on \(\phi \in \text{Hom}_R(S, M)\) by \(s \cdot \phi := \phi \circ s\), where we view \(s \in S\) as the “multiplication by \(s\)” map of \(S\).

We give \(\text{Hom}_R(F^e_* R, R)\) the structure of an \(F^e_* R\)-module in exactly this way. So an element \(F^e_* r\) acts on \(\phi \in \text{Hom}_R(F^e_* R, R)\) by precomposition with multiplication by \(F^e_* r\). That is, \(F^e_* r \cdot \phi\) is the composition \(\phi \circ F^e_* r\), which is the map

\[ F^e_* R \xrightarrow{F^e_* r} F^e_* R \xrightarrow{\phi} R \quad F^e_* x \mapsto F^e_* (rx) \mapsto \phi(F^e_*(rx)). \]

We examine this \(F^e_* R\)-module structure of \(\text{Hom}_R(F^e_* R, R)\) in a few cases:

**Lemma 5.1.** Suppose that \(k\) is an \(F\)-finite field, then \(\text{Hom}_k(F^e_* k, k) \cong F^e_* k\) as \(F^e_* k\)-vector spaces.
5. THE PROOF OF FEDDER’S CRITERION

**Proof.** Fix any non-zero \( \psi \in \text{Hom}_k(F^e_s k, k) \). We have an \( F^e_s k \)-linear map

\[
F^e_s k \longrightarrow \text{Hom}_k(F^e_s k, k) \quad F^e_s \lambda \mapsto \psi \circ F^e_s \lambda.
\]

This map is easily seen to be \( F^e_s k \)-linear, and non-zero (hence injective). On the other hand, it is also a map of \( k \)-vector spaces of the same dimension over \( k \), namely \( F^e_s k \) and its \( k \)-dual, so it must be surjective as well. \( \square \)

We next prove a similar result for polynomial and power series rings. Let us first fix some notation.

**Notation 5.2.** Fix an \( F \)-finite field \( k \), and let \( S \) be the polynomial ring \( k[x_1, \ldots, x_n] \), its localization at the maximal ideal \( (x_1, \ldots, x_n) \) or the power series ring \( k[[x_1, \ldots, x_n]] \).

Fix a basis \( \{F^e_s \lambda_i\} \) for \( F^e_s k \) over \( k \), and assume without loss of generality that \( F^e_s \lambda_1 = F^e_s 1 \).

Recall that \( F^e_s S \) is freely generated over \( S \) by the finite basis \( \{F^e_s(\lambda_i x^a)\} \) where \( x^a \) denotes the monomial of the form \( x_1^{a_1} \cdots x_n^{a_n} \) with \( 0 \leq a_i \leq p^e - 1 \), and the \( \lambda_i \) ranges through all the basis elements \( F^e_s \lambda_i \) in our chosen basis for \( F^e_s k \) over \( k \).

Thus as an \( S \)-module, \( \text{Hom}_S(F^e_s S, S) \) is freely generated by the dual basis to \( \{F^e_s(\lambda_i x^a)\} \), consisting of the projections

\[
(5.2.1) \quad F^e_s S \overset{\rho_{\lambda_i x^a}}{\longrightarrow} S
\]
on to the \( S \)-module summands of \( F^e_s S \) spanned by the basis elements \( F^e_s(\lambda_i x^a) \).

**Proposition 5.3.** With notation as above, \( \text{Hom}_S(F^e_s S, S) \) is free of rank one as a module over \( F^e_s S \), generated by the projection \( \Phi = \rho_{x^{p^e-1} a} \) of the free \( S \)-module \( F^e_s S \) onto the \( S \)-summand spanned by \( F^e_s(x_1 \cdots x_n)^{p^e - 1} \). Explicitly, \( \Phi \) is the \( S \)-module map defined on our free \( S \)-module basis \( \{F^e_s(\lambda_i x^a)\} \) by

\[
\Phi(F^e_s(\lambda_i x^a)) = \begin{cases} 
1 & \text{if } a_1 = \cdots = a_n = p^e - 1, \lambda_i = 1 \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** In order to prove that \( \text{Hom}_S(F^e_s S, S) \) is generated by \( \Phi \) as an \( F^e_s S \) module, it will suffice to show that each projection \( \rho \) in our dual basis \([5.2.1]\) for the \( S \)-module \( \text{Hom}_S(F^e_s S, S) \) can be obtained from \( \Phi \) by pre-composition with some element in \( F^e_s S \).

We first prove the Proposition in the special case that \( k \) is perfect (so the basis for \( F^e_s k \) over \( k \) is just \( F^e_s \lambda_1 = F^e_s 1 \)). Consider a projection \( \rho \) of the form \( \rho_{x^a} \). Denote by \( x^{p^e-1-a} \) the monomial whose exponent on \( x_i \) is \( p^e - 1 - a_i \); since \( 0 \leq a_i \leq p^e - 1 \), the monomial \( x^{p^e-1-a} \) also has all exponents between 0 and \( p^e - 1 \). We claim that \( \rho_{x^a} = \Phi \circ F^e_s(x^{p^e-1-a}) \). That is, we claim that the the composition

\[
F^e_s S \overset{F^e_s(x^{p^e-1-a})}{\longrightarrow} F^e_s S \overset{\Phi}{\longrightarrow} S
\]
is the projection \( \rho \). This can be verified by checking its effect on each of the free basis elements \( F_*(x^b) \) where \( x^b = x_1^{b_1} \ldots x_n^{b_n} \) with \( 0 \leq b_i \leq p^e - 1 \). Note that under the first map,

\[
F_*(x^b) \mapsto F_*(x^b)F_*(x^{p^e-1-a}) = F_*(x^{p^e-1-a+b})
\]

which is sent to zero under the second map unless \( a = b \), in which case it is sent to 1. Thus \( \rho = \Phi \circ F_*(x^{p^e-1-a}) \), as needed. Since each of the \( S \)-module generators is an \( F_*(S) \)-multiple of \( \Phi \), we see that \( \Phi \) is a generator for \( \text{Hom}_S(F_*(S), S) \) as a module over \( F_*(S) \).

The case where \( k \) is not perfect is only slightly more technical. To fix notation, say \( S = k[x_1, \ldots, x_n] \) (the case where we localize or complete can treated similarly). Note that we can factor the ring map \( S \xrightarrow{F_*} F_*(S) \) as

\[
k[x_1, \ldots, x_n] \xrightarrow{(F_*k)} [x_1, \ldots, x_n] \xrightarrow{F_*(k)} [x_1, \ldots, x_n]
\]

where the first extension is free with basis \( \{ F_*\lambda_i \} \) and the second is free with basis \( \{ x_1^{a_1} \ldots x_n^{a_n} \} \) where \( 0 \leq a_i \leq p^e - 1 \). Let \( \nu \) be the projection \( F_*k \rightarrow k \) onto the summand spanned by the basis element 1, and let \( \Phi' : F_*(k[x_1, \ldots, x_n]) \rightarrow (F_*k)[x_1, \ldots, x_n] \) be the projection onto the summand spanned by \( (x_1 \ldots x_n)^{p^e-1} \).

Note that \( \Phi \) is the composition

\[
F_*(S) \xrightarrow{\Phi'} (F_*k)[x_1, \ldots, x_n] \xrightarrow{\nu} S
\]

which is easily checked by examining what this composition does to the basis elements \( \{ F_*\lambda_i x^b \} \). Here, the map \( \Phi' \) is \( F_*k \)-linear and the map \( \nu \) is extended to the polynomial ring in the obvious way, so \( \nu \) restricts to the identity map on \( k[x_1, \ldots, x_n] \).

To check that \( \Phi \) generates \( \text{Hom}_S(F_*(S), S) \), we need to write each projection \( \rho \) of the form \( \rho_{\lambda x^a} \) as a composition \( \Phi \circ F_*(z) \) for some \( z \in S \). Since \( \nu \) generates \( \text{Hom}_k(F_*k, k) \), we can write the projection \( F_*k \rightarrow k \) onto \( \lambda \) as \( \nu \circ F_*b \) for some \( b \in k \). It is not hard to check that

\[
\rho_{\lambda x^a} = \nu \circ F_*b \circ \Phi' \circ F_*(x_1^{p^e-1-a_1} \ldots x_n^{p^e-1-a_n})
\]

by checking that they take the same value on every basis element. But also because \( \Phi' \) is \( F_*k \)-linear, this is

\[
\rho_{\lambda x^a} = (\nu \circ \Phi' \circ F_*b)(F_*(x_1^{p^e-1-a_1} \ldots x_n^{p^e-1-a_n})) = \Phi \circ F_*b(F_*(x_1^{p^e-1-a_1} \ldots x_n^{p^e-1-a_n}))
\]

Since each of the \( S \)-module generators is an \( F_*(S) \)-multiple of \( \Phi \), so \( \Phi \) generates \( \text{Hom}_S(F_*(S), S) \) as a module over \( F_*(S) \).

It remains only to show that the \( F_*(S) \)-linear map

\[
F_*(S) \rightarrow \text{Hom}_S(F_*(S), S) \quad F_*y \mapsto \Phi \circ F_*y
\]

is injective. Suppose it has non-zero kernel \( K \). Viewing \( K \) as an \( S \)-module, we see that \( K \) is torsion-free (it is a submodule of the free module \( F_*(S) \)). So tensoring with the fraction field of \( L \) of \( S \), the map

\[
F_*(S) \otimes_S L \rightarrow \text{Hom}_S(F_*(S) \otimes_S L) \quad F_*y \mapsto \Phi \circ F_*y
\]
also has non-zero kernel \( L \otimes_S K \). But this contradicts Lemma 5.1 since we know the map

\[
F^e_* L \to \text{Hom}_L(F^e_* L, L)
\]

\( F^e_* y \mapsto \Phi \circ F^e_* y
\)
is an isomorphism. The proof is complete. \( \square \)

**Remark 5.4.** The generator \( \Phi \) for \( \text{Hom}_S(F^e_* S, S) \) as an \( F^e_* S \)-module is sometimes called a **generating map**. Of course, \( \Phi \) depends on \( e \), so we write \( \Phi^e \) when there is any risk of confusion. Note that for an arbitrary monomial \( x^b \), we have

\[
\Phi(F^e_*(x_1^{b_1} \cdots x_n^{b_n})) = \left\{ \begin{array}{ll}
x_1^{\frac{b_1-(p^e-1)}{p^e}} \cdots x_n^{\frac{b_n-(p^e-1)}{p^e}} & \text{if } b_i \equiv p^e - 1 \pmod{p^e} \forall i \\
0 & \text{otherwise}
\end{array} \right.
\]

**Remark 5.5.** For any regular local F-finite ring \( S \), one can show that \( \text{Hom}_S(F^e_* S, S) \) is free of rank one as an \( F^e_* S \)-module. Indeed, this is true more generally—for any Gorenstein local F-finite ring. We won’t need this here, but we explain why for those who have the background. The point is that if \( R \to S \) is a finite map and \( R \) has a canonical module \( \omega_R \), then \( S \) has canonical module \( \omega_S \cong \text{Hom}_R(S, \omega_R) \) (Grothendieck calls this \( f^! \omega_R \)). Now if \( S \) is a local Gorenstein ring, then by definition of Gorenstein \( \omega_S \cong S \). Since \( \text{Hom}_S(F^e_* S, S) \) is a canonical module for \( F^e_* S \) and \( F^e_* S \) is also Gorenstein, it follows that \( \text{Hom}_S(F^e_* S, S) \cong F^e_* S \).

### 5.2. The \( F^e_* R \)-module \( \text{Hom}_R(F^e_* R, R) \)

Consider a ring \( R = S/I \) where \( S \) is regular and F-finite. We want to relate the maps \( \text{Hom}_R(F^e_* R, R) \) to the maps \( \text{Hom}_S(F^e_* S, S) \).

Given any \( R \)-module map \( F^e_* R \to R \), consider the following commutative diagram of \( S \)-modules:

\[
\begin{array}{ccc}
F^e_* S & \xrightarrow{\tilde{\psi}} & S \\
\downarrow & & \downarrow \\
F^e_* R & \xrightarrow{\tilde{\psi}} & R.
\end{array}
\]

The dotted arrow \( \tilde{\psi} \) exists because, by Kunz’s Theorem, \( F^e_* S \) is a projective \( S \)-module (although \( \tilde{\psi} \) is not unique). Thus every map \( \psi \) in \( \text{Hom}_R(F^e_* R, R) \) is induced by a map \( \tilde{\psi} \) in \( \text{Hom}_S(F^e_* S, S) \). Because \( \tilde{\psi} \) lifts \( \psi \), it necessarily satisfies

\( \tilde{\psi}(F^e_* I) \subset I \).
Theorem 5.6 (Fedder’s Lemma). Let $S$ be an arbitrary $F$-finite ring, and let $R = S/I$. Then there is a natural $F^e_r S$-module map

$$\left( F^e_r (I^{[p^r]} : I) \right) \cdot \text{Hom}_S(F^e_r S, S) \rightarrow \text{Hom}_R(F^e_r R, R).$$

When $S$ is regular, the map $\Psi$ is surjective with kernel $F^e_r (I^{[p^r]} : I) \cdot \text{Hom}_S(F^e_r S, S)$.

That is, if $R = S/I$ where $S$ is an $F$-finite regular ring, then

$$\text{Hom}_R(F^e_r R, R) \cong \frac{\left( F^e_r (I^{[p^r]} : I) \right) \cdot \text{Hom}_S(F^e_r S, S)}{F^e_r I^{[p^r]} \cdot \text{Hom}_S(F^e_r S, S)}.$$

Remark 5.7. The notation $\left( F^e_r (I^{[p^r]} : I) \right) \cdot \text{Hom}_S(F^e_r S, S)$ has been known to cause some confusion. The point is that we are viewing $\text{Hom}_S(F^e_r S, S)$ as a module over the ring $F^e_r S$. For any ideal $J \subset S$, including the ideal $J = (I^{[p^r]} : I)$, there is a corresponding ideal $F^e_r J$ of $F^e_r S$, as well as a corresponding submodule $F^e_r J \text{Hom}_S(F^e_r S, S)$ of $\text{Hom}_S(F^e_r S, S)$. In the statement of Theorem 5.6 there are two appearances of such a $F^e_r S$-module, one with $J = (I^{[p^r]} : I)$ and one with $J = I^{[p^r]}$.

The next lemma will be used in the proof of Fedder’s Lemma:

Lemma 5.8. Let $S$ be an $F$-finite regular ring, and let $\Phi \in \text{Hom}_S(F^e_r S, S)$ be an $F^e_r S$-module generator. Let $I$ and $J$ be arbitrary ideals of $S$. Then

$$\Phi(F^e_r J) \subset I \Leftrightarrow J \subset I^{[p^r]}.$$

Proof. First suppose $J \subset I^{[p^r]}$. Then clearly

$$\Phi(F^e_r J) \subset \Phi(F^e_r I^{[p^r]}) = I\Phi(F^e_r S) \subset I.$$

This actually holds for arbitrary $\Phi \in \text{Hom}_S(F^e_r S, S)$, not just the generator.

For the converse, first note that there is no loss of generality in assuming $S$ is local, or even complete, so we assume that $F^e_r S$ is free over $S$ and that $\Phi$ is an $F^e_r$-module generator for $\text{Hom}_S(F^e_r S, S)$.

Suppose that the free $S$-module $F^e_r S$ has basis $\{F^e_r g_i\}_{i=1}^d \subset F^e_r S$. Let $\pi_1, \ldots, \pi_d$ denote the corresponding dual basis for $\text{Hom}_S(F^e_r S, S)$. By definition of the dual basis, the $S$-linear map

$$F^e_r S \rightarrow F^e_r S$$

$$F^e_r g_1 \pi_1 + F^e_r g_2 \pi_2 + \cdots + F^e_r g_t \pi_t$$

is the identity map on $F^e_r S$.

Now to prove the converse, assume that $\Phi(F^e_r J) \subset I$. Because $\Phi$ is a generator, the projections $\pi_i \in \text{Hom}_S(F^e_r S, S)$ can be written $\pi_i \circ F^e_r r_i$ for some $r_i \in S$. Thus

$$\pi_i(F^e_r J) = (\Phi \circ F^e_r r_i)(F^e_r J) = \Phi(F^e_r (r_i J)) \subset \Phi(F^e_r J) \subset I$$
for all \(\pi_i\). But since \(\sum_{i=1}^{t} F_{s}^{e} g_i \pi_i\) is the identity map on \(F_{s}^{e} S\), applying it to \(F_{s}^{e} J \subset F_{s}^{e} S\), we have

\[
F_{s}^{e} J = \sum_{i=1}^{t} F_{s}^{e} g_i \pi_i (F_{s}^{e} J) \subset IF_{s}^{e} S = F_{s}^{e}(I^{[p^r]}).
\]

This means that \(J \subset I^{[p^r]}\).

**Proof of Theorem 5.6.** First, suppose that \(u \in (I^{[p^r]} : I)\) and \(\phi \in \text{Hom}_S(F_{s}^{e} S, S)\). We need to check that the composition \(F_{s}^{e} S \xrightarrow{F_{s}^{e} u} F_{s}^{e} S \xrightarrow{\phi} S\) descends to a well-defined \(R\)-linear map

\[
F_{s}^{e} R \rightarrow R.
\]

For this, we need that \(\psi(F_{s}^{e}(I)) \subset I\). Checking this, we have

\[
\psi(F_{s}^{e} I) = (\phi \circ F_{s}^{e} (u)) (F_{s}^{e} I) = \phi(F_{s}^{e} (u I)) \subset \phi(F_{s}^{e} (I^{[p^r]})) \subset I \phi(F_{s}^{e} S) \subset I,
\]

so that \(\psi\) does indeed determine a well-defined map \(F_{s}^{e} R \rightarrow R\).

It remains to check that when \(S\) is regular, the map \(\Psi\) is surjective and has the desired kernel. Both of these conditions can be checked locally, since the formation of the quotient \(S/I\), the modules \(\text{Hom}_S(F_{s}^{e} S, S)\) and \(\text{Hom}_R(F_{s}^{e} R, R)\), and the ideal \((I^{[p^r]} : I)\) commute with localization at any prime of \(S\). So there is no loss of generality in assuming that \(S\) is local. Likewise, the formation of the quotient \(S/I\), the modules \(\text{Hom}_S(F_{s}^{e} S, S)\) and \(\text{Hom}_R(F_{s}^{e} R, R)\), and the ideal \((I^{[p^r]} : I)\) commute with the faithfully flat base change \(S \rightarrow \hat{S}\), where \(\hat{S}\) is the completion of \(S\) at the unique maximal ideal. So without loss of generality, we assume that \((S, \mathfrak{m})\) is a complete local regular ring—by Cohen’s Structure Theorem, we can assume that \(S = k[[x_1, \ldots, x_n]]\).

Now to check that \(\Psi\) is surjective: recall that because \(F_{s}^{e} S\) is projective, any map \(\psi\) in \(\text{Hom}_R(F_{s}^{e} R, R)\) lifts to a map \(\tilde{\psi}\) in \(\text{Hom}_S(F_{s}^{e} S, S)\). This \(\tilde{\psi}\) necessarily satisfies

\[
\tilde{\psi}(F_{s}^{e}(I)) \subset I.
\]

Using the fact that \(\Phi\) is a \(F_{s}^{e} S\)-module generator for \(\text{Hom}_S(F_{s}^{e} S, S)\), write

\[
\tilde{\psi} = \Phi \circ F_{s}^{e} y
\]

for some \(y \in S\) (Proposition 5.3). We need to show that \(y \in (I^{[p^r]} : I)\).

Since \(\tilde{\psi}(F_{s}^{e}(I)) \subset I\), we have

\[
(\Phi \circ F_{s}^{e}(y))(F_{s}^{e}(I)) = \Phi(F_{s}^{e}(y I)) \subset I.
\]

Lemma 5.8 now implies that

\[
y I \subset I^{[p^r]},
\]

or \(y \in (I^{[p^r]} : I)\), as needed. This completes the proof that when \(S\) is regular, the map \(\Psi\) is surjective.
Finally, we consider the kernel of $\Psi$. Take arbitrary 
\[ \phi = \Phi \circ F^e y \in \left( F^e_* (I^{[p^e]} : I) \right) \cdot \text{Hom}_S(F^e_* S, S) \]
in the kernel of $\Psi$. This means in particular that 
\[ \phi(F^e_* S) = (\Phi \circ F^e y)(F^e_* S) = \Phi(F^e(y)) \subset I. \]
So Lemma 5.8 implies that $y \in I^{[p^e]}$. This shows that kernel of $\Psi$ is $F^e_* (I^{[p^e]}) \text{Hom}_S(F^e_* S, S)$. 

We have done the hard work needed to prove Fedder’s Criterion. Now we need only reap what we’ve sown. We will prove the following slightly stronger theorem that implies Fedder’s Criterion, Theorem 5.9.

**Theorem 5.9.** Suppose that $S$ is an $F$-finite regular ring and $R = S/I$. Then $R$ is Frobenius split in a neighborhood of a prime ideal $q \in V(I) \subseteq \text{Spec } S$ if and only if (for any fixed $e$)
\[ (I^{[p^e]} : I) \not\in q^{[p^e]}. \]

**Proof.** To check the statement, we can localize and even complete at $q$ with out loss of generality.

Suppose that $R$ is $F$-split in a neighborhood of a prime ideal $q \in V(I)$. It follows that the evaluation-at-$1$ map $\text{Hom}_R(F^e_* R, R) \to R$ surjects in a neighborhood of $q$. In particular, there is some $\phi_R \in \text{Hom}_R(F^e_* R, R)$ such that $\phi_R(F^e_* 1) \not\in q/I$. It follows from Theorem 5.6 that $\phi_R$ is induced by some $\phi_S = \Phi \circ F^e y \in \text{Hom}_S(F^e_* S, S)$, where $y \in (I^{[p^e]} : I)$. That is,
\[ \phi_S(F^e_* 1) = \Phi(F^e y) \not\in q. \]

Suppose for a contradiction now that $(I^{[p^e]} : I) \subseteq q^{[p^e]}$. In this case, 
\[ \phi_S(F^e_* 1) = \Phi(F^e y) \in \Phi\left( F^e_* (I^{[p^e]} : I) \right) \subset \Phi\left( F^e_* q^{[p^e]} \right) \subset q \Phi(F^e_* S) \subset q, \]
but this contradicts $\phi_S(F^e_* 1) \not\in q$.

Conversely, suppose that $b \in (I^{[p^e]} : I) \setminus q^{[p^e]}$. Let $\Phi \in \text{Hom}_S(F^e_* S, S)$ be a generating homomorphism, and write $\phi_S = \Phi \circ F^e b$. Since $b \not\in q^{[p^e]}$, Lemma 5.8 implies that $\Phi(F^e(b)) \not\in q$. In particular, $\Phi(F^e b) \not\in q$, and so $\phi(F^e_* 1) \not\in q$. This means that the localization of the map $\phi$ at $q$ induces a map
\[ F^e_* R_q \stackrel{\phi}{\longrightarrow} R_q, \]
sending $F^e_* 1$ to a unit. This is enough to prove that $R_q$ is Frobenius split. 

**Exercise 5.1.** Prove that that hypersurface $xy - z^2$ in $\mathbb{A}^3$ is Frobenius split in every characteristic.

**Exercise 5.2.** Describe all $f \in \mathbb{F}_2[x, y, z]$ such that $\mathbb{F}_2[x, y, z]/(f)$ is Frobenius split.
5.3. The Locus of Non-Frobenius Split Points. We have already proved that the locus of non-Frobenius split points in an $F$-finite ring is a closed set. Fedder’s lemma gives us a very explicit way to compute an ideal defining this locus.

**Theorem 5.10.** Suppose that $S$ is an $F$-finite regular ring and $R = S/I$. Let $J_e \subseteq S$ denote the image of the evaluation-at-1 map

\[
\text{Image} \left( \left( F^e_*(I^{[p^e]} : I) \right) \cdot \text{Hom}_S(F^e_*S, S) \rightarrow S \right)
\]

for some integer $e > 0$. Then the set theoretic locus $V(J_e) \subseteq V(I) \subseteq \text{Spec } S$ is the set of points of $V(I) \cong \text{Spec } R$ where $\text{Spec } R$ is not $F$-split.

Before proving this result, we notice that the result implies that $V(J_e)$ is independent of $e$. However, scheme theoretically, $V(J_e)$ is generally not independent of $e$.

**Proof.** The evaluation-at-1 map in the statement of the theorem is surjective at all points $q \in V(I) \subseteq \text{Spec } S$ where $\text{Hom}_R(F^e_*R, R) \rightarrow R$ is also surjective. Of course, outside of $V(I)$, $(I^{[p^e]} : I)$ agrees with $S$ and the surjectivity is obvious. The result follows since the iterated Frobenius map $R_q \rightarrow F^e_!R_q$ splits if and only if the Frobenius map $R_q \rightarrow F^e_*R_q$ splits.

**Corollary 5.11.** The locus where $\text{Spec } R$ is not split is closed and it is equal to $V\left( \Phi^e(F^e_*(I^{[p^e]} : I)) \right)$.

**Remark 5.12.** The ideal $\Phi^e(F^e_*(I^{[p^e]} : I))$ depends on the choice of $e$, although the locus it defines does not!

**Exercise 5.3.** Show that $\Phi^e(F^e_*(I^{[p^e]} : I)) \supseteq \Phi^{e+1}(F^{e+1}_*(I^{[p^{e+1}]} : I))$.

**Hint:** Show that $\Phi^e(F^e_*(I^{[p^e]} : I)) \cdot R$ is the same as the image of the evaluation-at-1 map $\text{Hom}_R(F^e_*R, R) \rightarrow R$.

**Question 5.13** (Open question). It is an open question whether the descending ideals from the previous exercise stabilize (are all equal for $e \gg 0$). This is known if $R$ is a hypersurface or more generally Gorenstein or even more generally $Q$-Gorenstein. The Gorenstein case is essentially a key step in a famous result of Hartshorne and Speiser [HS77].

**Exercise 5.4.** Suppose that $R$ is a regular Noetherian ring of characteristic $p > 0$ and that $q$ is a prime ideal. Prove that $q^{[p^e]}$ is $q$-primary.

**Hint:** Show that if $f \notin q$, then $0 \rightarrow R/q^{[p^e]} \xrightarrow{f} R/q^{[p^e]}$ injects.
CHAPTER 5

Log Resolutions and F-singularities for Pairs

In the 1980’s, the theory of pairs took center stage in birational geometry. Two decades later, characteristic \( p \) commutative algebra began to develop for pairs, and eventually, a full theory of F-singularities of pairs blossomed.

In the original and most basic setting, a “pair” \((X, \Delta)\) is a smooth variety \(X\) together with an effective divisor (or \(\mathbb{Q}\)-divisor) \(\Delta\) on it. More technical generalizations eventually led to the definition of a log pair \((X, \Delta)\), where \(X\) is normal and \(\Delta\) is an effective \(\mathbb{Q}\)-Weil divisor with the property that \(\Delta + K_X\) is \(\mathbb{Q}\)-Cartier. This means that \(\Delta\) is a non-negative \(\mathbb{Q}\)-linear combination of irreducible codimension one closed subvarieties of \(X\) and that for some positive integer \(m\), the Weil divisor \(m(\Delta + K_X)\) is locally principle (Cartier), and therefore can be pulled back under any morphism to \(X\). Singularities of Pairs \((X, \Delta)\) for complex varieties are an important technical component in the minimal model program. F-singularities of pairs play a corresponding role in prime characteristic.

In this section, we will discuss log resolutions of pairs, first in the classical setting of a smooth ambient variety. We will define singularities of pairs and the log canonical threshold before moving on to F-splitting and F-regularity of pairs and the F-pure threshold.

1. Log Resolutions

Roughly speaking, a log resolution of a log pair \((X, \Delta)\) is a proper birational map \(Y \to X\) with \(Y\) smooth such that

(a) the pull-back of \(\Delta\) is as nice a possible (its support is a union of smooth divisors intersecting transversely)

(b) the exceptional set of \(\pi\) is also as nice as possible (again, normal crossing support), and

(c) the pullback of \(\Delta\) and the exceptional set intersect as nicely as possible (transversely).

Such a log resolution always exists by Hironaka’s theorem on resolution of singularities, although we need to explain more carefully how to make sense of all this.

Let us recall what it means that an effective divisor \(D\) on a smooth variety \(Y\) has **normal crossing support**. Briefly, a reduced divisor \(D = \sum_{i=1}^t D_i\) has normal crossing support if each component \(D_i\) is smooth, and the \(D_i\) intersect transversely.
Two divisors $D_1$ and $D_2$ intersect transversely if their (scheme theoretic intersection) is a smooth codimension two subscheme of $Y$; more generally, divisors $D_1, \ldots, D_n$ intersect transversely if the scheme theoretic intersection $D_1 \cap D_2 \cdots \cap D_n$ is smooth of codimension $n$ in $Y$.

Alternatively, normal crossing divisors can be understood in terms of local coordinates on $Y$ as follows. Recall that for each point $y$ of the regular scheme $Y$, the local ring $\mathcal{O}_{Y,y}$ is regular local ring. Its maximal ideal $m$, therefore, is generated by $x_1, x_2, \ldots, x_d$ where $d$ is the dimension of the local ring. The images of these element in $m/m^2$ are a basis for this Zariski cotangent space. Of course, the generators for $m$ are not unique: any basis of $m/m^2$ lifts to minimal set of generators for $m$. Such a minimal set of generators for $m$ give us local coordinates for $Y$ at $y$.

Now, suppose the point $y$ is in the support of the divisor $D$. This means that the defining equation, $f$, of $D$ in a neighborhood of $y$ is in the maximal ideal $m$ at $y$. Note that $D$ is smooth at $y$ if and only if $f \in m \setminus m^2$—that is, if and only if $f$ is part of a minimal generating set of $m$. This is because the local ring $\mathcal{O}_{D,y} \cong \mathcal{O}_{Y,y}/(f)$ is regular if and only if $f \in m \setminus m^2$. An effective reduced divisor $D = \sum_{i=1}^n D_i$ is therefore in normal crossings at $y$ if the components $D_1, \ldots, D_n$ passing through $y$ have defining equations $x_1, \ldots, x_n$ that are part of a minimal set of generators for the ideal $m$ of $y$.

To summarize: An effective divisor $D$ on a smooth variety $Y$ can be succinctly defined as having normal crossing support if, at every point $y \in Y$, the defining equation of $D$ at $y$ is a monomial in local coordinates at $y$. Of course, if $y$ is not in the support of $D$, then the defining equation for $D$ at $y$ can be taken to be $1 \in \mathcal{O}_{Y,y}$, which can be interpreted as the trivial monomial $x_1^0 \cdots x_d^0$.

1.0.1. Log Resolution for a divisor on a Smooth Ambient Variety. The original and still most central case to understand is the case when the ambient variety $X$ of the pair $(X, \Delta)$ is a smooth variety.

When $X$ is smooth, a log resolution $Y \xrightarrow{\pi} X$ can be described as a change of coordinates such that both $\pi^* \Delta$ and the Jacobian determinant are locally monomial in the same coordinates. Indeed, the Jacobian determinant (of the matrix of partial derivatives in local coordinates for the map $Y \xrightarrow{\pi} X$ between smooth $n$-dimensional varieties) is obviously supported precisely on the exceptional divisor, since at any point where $\pi$ is an isomorphism, the Jacobian determinant is necessarily a unit.

Example 1.1. Consider the irreducible divisor $D$ in $\mathbb{A}^3$ defined $x^2 + y^2 - z^2 = 0$. The divisor $D$ has an isolated singularity at the origin of $\mathbb{A}^3$. Let $\pi : X \to \mathbb{A}^3$ be the blow up of the origin. The variety $X$ is covered by three affine charts isomorphic to $\mathbb{A}^3$. We consider the chart whose coordinates are $x_1 = \frac{x}{z}, y_1 = \frac{y}{z}$ and $z_1 = z$. The map $\pi$ on this chart is $\mathbb{A}^3 \to \mathbb{A}^3 \quad (x_1, y_1, z_1) \mapsto (x_1 z_1, y_1 z_1, z_1)$.

Note that the divisor $E = \{z_1 = 0\}$ is the full set of points in this chart which are mapped to $(0, 0, 0)$—that is, the exceptional divisor of $\pi$ is defined by $z_1$ in this chart.
The divisor $D$ pulls back, on this chart, to the divisor defined by the pullback of the function $x^2 + y^2 - z^2$ under $\pi$, namely to the divisor defined by $(x_1 z_1)^2 + (y_1 z_1)^2 - z_1^2 = z_1^2 (x_1^2 + y_1^2 - 1) = 0$. Note that $\pi^* D$ has two irreducible components: one is the cylinder $\tilde{D}$ defined by the vanishing of $x_1^2 + y_1^2 - 1$ (the proper transform of $D$ under $\pi$), the other is $2E$ where $E$ is the exceptional divisor. Because the intersection of $E$ and $\tilde{D}$ is a circle, which is smooth and of codimension two in $\mathbb{A}^3$, the components of $\pi^* D$ intersect transversely. That is, $\pi^* D = \tilde{D} + 2E$ has normal crossing support.

Let us compute the Jacobian determinant of this change of coordinates:

$$\det \begin{pmatrix} z_1 & 0 & x_1 \\ 0 & z_1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} = z_1^2.$$ 

We use the notation $K_\pi$ to denote the divisor defined by this Jacobian determinant, namely $2E$ in this case. Alternatively, we can compute the Jacobian determinant by comparing $\pi^* (dx \wedge dy \wedge dz) = d(x_1 z_1) \wedge d(y_1 z_1) \wedge dz = z_1^2 dx_1 \wedge dy_1 \wedge dz_1,$ which is a generator for $\pi^* \omega_\mathbb{A}^3$, to the generator $dx_1 \wedge dy_1 \wedge dz_1$ for $\omega_X$ on the chart in question. We see that the “difference” between these volume forms is $z_1^2$, the Jacobian determinant, which defines the effective divisor $2E$.

**Definition 1.2.** Let $\pi : Y \to X$ be a proper birational morphism between smooth varieties. The relative canonical divisor of $\pi$, denoted $K_\pi$ or $K_{Y/X}$, is the effective divisor on $Y$ defined locally by the Jacobian determinant. To understand this definition, fix a proper birational morphism $\pi : Y \to X$ between smooth varieties. Differential forms on $X$ can be pulled back under $\pi$, giving a natural inclusion

$$\pi^* \omega_X \subset \omega_Y.$$ 

This map is the identity map on the locus where $\pi$ is an isomorphism, so that the cokernel is supported on the exceptional set of $\pi$. Tensoring with the invertible sheaf $(\pi^* \omega_X)^{-1}$, we have

$$\mathcal{O}_Y \subset (\pi^* \omega_X)^{-1} \otimes \omega_Y,$$ 

so that the “difference” between the canonical modules on $X$ and $Y$ is the invertible sheaf $(\pi^* \omega_X)^{-1} \otimes \omega_Y$ defined locally by the Jacobian determinant. We call this invertible sheaf the relative canonical sheaf and denote it by $\omega_{Y/X}$. We can define the relative canonical divisor as the unique exceptionally supported divisor $K_\pi$ so that $\mathcal{O}_Y (K_\pi) = \omega_{Y/X}$. Clearly

$$K_\pi \equiv K_Y - \pi^* K_X,$$ 

Note that since

$$\mathcal{O}_Y \subset \mathcal{O}_Y (K_\pi),$$ 

the relative canonical divisor $K_\pi$ is an effective divisor supported precisely on the exceptional locus. If we choose a divisor $K_Y$ such that $\omega_Y \equiv \mathcal{O}_Y (K_Y)$ and let $K_X = \pi_* K_Y$, then it is not hard to see that $\mathcal{O}_X (K_X) = \omega_X$. Clearly

$$K_\pi \equiv K_Y - \pi^* K_X.$$
(where $\equiv$ denotes linear equivalence) and in fact, $K_\pi$ is the unique exceptionally supported effective divisor in the linear class $|K_Y - \pi^*K_X|$. [Here, it is is important to remember that $\pi$ is a map between smooth varieties; when we later make sense of $K_\pi$ in more general cases, we will need to be cautious.]

**Proposition 1.3.** Let $\pi : Y \to X$ be the blow up of a smooth subvariety $X$ along a smooth subvariety $Z$. Then the relative canonical divisor $K_\pi$ is $(c-1)E$, where $E$ is the exceptional divisor and $c$ is the codimension of $Z$ in $X$.

**Proof.** We provide a sketch, leaving the details as an exercise. Fix a point $x$ of $Z \subset X$ and let $x_1, \ldots, x_d$ be local coordinates in a neighborhood of some point such that $Z$ is defined by $x_1, \ldots, x_c$. Consider the blowup $Y$ of $X$ along $Z$: it is covered by charts whose local coordinates, in one of these, are $x_1, \frac{x_2}{x_1}, \ldots, \frac{x_c}{x_1}, x_{c+1}, \ldots, x_d$, which we label $y_1, \ldots, y_d$. The form $dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_d$ pulls back to

$$\pi^*(dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_d) = dy_1 \wedge d(y_1y_2) \wedge \cdots \wedge d(y_1y_c) \wedge dy_{c+1} \wedge \cdots \wedge dy_d$$

$$= y_1^{c-1} dy_1 \wedge dy_2 \wedge \cdots \wedge dy_c \wedge dy_{c+1} \wedge \cdots \wedge dy_d.$$

The computation is similar in any chart, showing that $K_\pi = (c-1)E$. \qed

**Corollary 1.4.** Let $Y \xrightarrow{\pi} X \xrightarrow{\phi} W$ proper birational maps between smooth varieties. Suppose that $\pi$ is the blow up of a smooth variety $X$ along a smooth codimension $c$ subvariety $Z$. Then

$$K_{Y/W} = \pi^*(K_{X/W}) + (c-1)E$$

where $E$ is the exceptional divisor.

**Remark 1.5.** Suppose that $X$ is a normal variety of dimension $d$. Then we can define $\omega_X$ as the unique reflexive sheaf agreeing with $\Lambda^d\Omega_X$ on the smooth locus of $X$. If $\pi : Y \to X$ is a proper birational map, the sheaf $\pi^*\omega_X$ does not pull back nicely—it can have torsion, and may not be contained in $\omega_Y$, so we can not define a Jacobian or relative canonical divisor. However, if $\omega_X$ is invertible—which is the case if $X$ is Gorenstein—then we can pull back $\omega_X$ and compare it to $\omega_Y$. Put differently, if the Weil divisor $K_X$ is Cartier, we can define the relative canonical sheaf as the unique exceptional divisor in $K_Y - \pi^*K_X$. More generally, when $K_X$ is $Q$-Cartier—meaning that $mK_X$ is Cartier for some positive $m$, then we can define $K_\pi$ as $K_Y - \frac{1}{m}\pi^*(mK_X)$ although it no longer makes sense to say that its local defining equation is a Jacobian determinant. We will return to some of these generalization later.

There are even further generalizations that we will not treat here. For example, we can replace the $Q$ divisor by a formal combination of ideal sheaves.
Example 1.6. Let us compute a log resolution for the cuspidal curve \( D = \{ y^2 - x^3 = 0 \} \) in the plane \( \mathbb{A}^2 \). Because \( D \) has a singular point at the origin, we must blow it up. Let

\[ X_1 \xrightarrow{\pi_1} \mathbb{A}^2 \]

be the blowup up the origin in \( \mathbb{A}^2 \), with exceptional divisor \( E_1 \). Note that \( \pi_1^*D = \tilde{D} + 2E_1 \) and that in the local chart whose coordinates are

\[ x_1 = x, y_1 = \frac{y}{x}, \]

the divisor \( \pi_1^*D = \tilde{D} + 2E_1 \) is given by

\[ \pi_1^*(y^2 - x^3) = (x_1y_1)^2 - (x_1)^3 = x_1^2(y_1^2 - x_1). \]

This is the union of a smooth curve (a “parabola”) and a line (the exceptional divisor \( E_1 \)) but unfortunately, they do not intersect transversely: the parabola is tangent to the line. We need to blow up the non-transverse intersection point, the “origin” in this chart. We thus have

\[ X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbb{A}^2, \]

with \( E_2 \) denoting the new exceptional divisor. Abusing notation slightly, we write \( D \) and \( E_1 \) respectively for the proper transforms of \( D \) and \( E_1 \) on \( X_2 \), so that pulling back \( D \subset \mathbb{A}^2 \) we get

\[ \pi_2^*\pi_1^*(D) = (\tilde{D} + E_2) + 2(E_1 + E_2) = \tilde{D} + 2E_1 + 3E_2. \]

Examining this divisor in the chart whose coordinates are

\[ x_2 = \frac{x_1}{y_1}, y_2 = y_1, \]

we see that it is defined by

\[ \pi_2^*\pi_1^*(y^2 - x^3) = \pi_2^*(x_1^2y_2^2 - x_1^3) = (x_2y_2)^2(y_2^2) - (x_2y_2)^3 = x_2y_3^2(y_2 - x_2) \]

which is a union of three lines crossing at the origin. This is not quite yet in normal crossings: we must blow up one more time to separate out the three lines. The composition

\[ X_3 \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbb{A}^2, \]

where \( \pi_3 \) is the blowup at the origin in the relevant chart (whose exceptional divisor is \( E_3 \)) is a log resolution. Call this composition \( \pi \). We have that

\[ \pi^*D = D + 2E_1 + 3E_2 + 6E_3. \]

We compute the relative canonical divisor using Corollary [1.4]

\[ K_\pi = E_1 + 2E_2 + 4E_4. \]

Clearly, the divisor \( \pi^*D + K_\pi \) has normal crossing support.
5. Log Resolutions and F-Singularities for Pairs

2. Log Canonical Threshold

We would like to measure the singularity of a divisor $D$ defined by a complex polynomial $f$ on $\mathbb{C}^n$. The idea is to look at integrability of the function

$$\mathbb{C}^n = \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad z \mapsto \frac{1}{|f|^2}.$$ 

This function is not defined at the points of $D$, and indeed, as we approach $D$ along a path in $\mathbb{C}^n \setminus D$, the values of this function approach infinity (the function “blows up.”) At highly singular points—for example, at points of high multiplicity, where $f \in m_p^N$, $N \gg 0$—this function approaches infinity even faster. Thus the function is more likely to be integrable if we “dampen it” by raising to some small power $c$:

$$\mathbb{C}^n = \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad z \mapsto \frac{1}{|f|^{2c}}.$$ 

**Definition 2.1.** The log canonical threshold of $D$ (or of the function $f$) at a point $p \in \mathbb{C}^n$ is

$$\sup \{ c \in \mathbb{R}_>0 \mid \int_{B_p} \frac{1}{|f|^{2c}} < \infty \}$$

where $B_p$ is a small ball around the point $p$.

**Example 2.2.** Let us compute the log canonical threshold of the function $z^n$ at the origin. We compute $\int_{B_1} \frac{1}{|z|^{2nc}}$ in polar coordinates, remembering that $|z|^2 = r^2$ and $dx \wedge dy = r dr d\theta$. So

$$\int_{B_1} \frac{1}{|z|^{2nc}} = \lim_{\epsilon \to 0} \int_0^{2\pi} \int_\epsilon^R \frac{rd\theta}{r^{2nc}} = 2\pi \lim_{\epsilon \to 0} \int_\epsilon^R \frac{d\theta}{r^{2-2nc}} \left[ R^{2-2nc} - \epsilon^{2-2nc} \right]$$

which converges if and only if $2 - 2nc \geq 0$. That is, the integral converges if and only $c \leq \frac{1}{n}$. Taking the supremum, we see that the log canonical threshold of $z^n$ (at the origin) is $\frac{1}{n}$.

**Exercise 2.1.** As a direct generalization (using Fubini’s theorem), compute that the log canonical threshold of $z_1^{a_1} \cdots z_n^{a_n}$ is the minimum of the $\frac{1}{a_i}$.

**Theorem 2.3.** The log canonical threshold of $f \in \mathbb{C}[x_1, \ldots, x_n]$ at a point $p$ is a positive rational number.

We will prove this theorem using Hironaka’s theorem on log resolution. Note that it is not even obvious, from the definition, that the log canonical threshold is finite, nor non-zero, let alone rational.

We will also make use of the following lemma relating the real and complex Jacobians of a holomorphic map, whose proof we leave as an exercise:
Lemma 2.4. Let $\pi : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map, given coordinate-wise by the holomorphic functions $f_1, \ldots, f_n$ of $n$ variables. The holomorphic Jacobian $J_C(\pi)$ of $\pi$ is the determinant of the $n \times n$ matrix
\[
\frac{\partial f_i}{\partial z_j};
\]
not that $J_C$ is some holomorphic function. The real Jacobian of $\pi$ is the determinant of the $2n \times 2n$ matrix of partial derivatives for the same map $\pi$, but considered as a smooth map $\mathbb{R}^{2n} \xrightarrow{\pi} \mathbb{R}^{2n}$; note that $J_R(\pi)$ is a smooth real valued function. Then
\[
|J_C(\pi)|^2 = J_R(\pi).
\]

Example 2.5. The map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ can also be written
\[
\mathbb{R}^2 \to \mathbb{R}^2 \quad (x, y) \mapsto (x^2 - y^2, 2xy).
\]
The real jacobian is
\[
\det \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = 4(x^2 + y^2).
\]
The complex Jacobian is $2z$, so indeed
\[
|J_C|^2 = 4|z|^2 = 4(x^2 + y^2) = J_R.
\]

**Proof of Theorem 2.3.** Hironaka’s theorem guarantees the existence of a proper birational change of coordinates map $X \xrightarrow{\pi} \mathbb{C}^n$, with $X$ smooth, such that $\pi^*D + K_\pi$ has normal crossing support. This means that at every point of $X$, there are local coordinates (minimal generators for $m$) $x_1, \ldots, x_n$ such that both $f \circ \pi$ and the holomorphic Jacobian are given by monomials in $x_1, \ldots, x_n$ (up to unit multiple).

To compute $\int_{B_p} \frac{1}{|f|^2} \ dVol_{\mathbb{C}^n}$, where $B_p$ is a small ball around $p$, we pull back under $\pi$ to get
\[
\int_{\pi^{-1}(B_p)} \frac{1}{|f \circ \pi|^2} \pi^* \ dVol_{\mathbb{C}^n} = \int_{\pi^{-1}(B_p)} \frac{1}{|f \circ \pi|^2} J_R \ dVol_X
\]
where $J_R = |J_C|^2$ denotes the real Jacobian of $\pi$ as a map between real 2n-manifolds. At each point of $X$, both $f \circ \pi$ and $J_C$ are monomial in the same local coordinates $x_1, \ldots, x_n$ (up to unit multiple). Thus at each point of $\pi^{-1}(p)$, there is an open set $U$ where coordinates centered at $p$ can be used to compute the integral in that set:
\[
\int_{\pi^{-1}(B_p) \cap U} \frac{|J_C|^2}{|f \circ \pi|^2} \ dVol_X = \int_{U} \frac{|x_1^{c_1} \cdots x_n^{c_n}|^2}{|x_1^{b_1} \cdots x_n^{b_n}|^2} \ dVol_X.
\]
This is again a monomial integral similar to the one we computed before. In particular, it converges (as you should compute) if and only if $c \leq \frac{k_{b+1}}{b+1}$.

The set $\pi^{-1}(B_p)$ can be covered by open sets $U$ of the type above, where there are coordinates at each point of $\pi^{-1}(p)$ such that $\pi^*D + K_\pi$ is monomial in them as above. The integral $\int_{\pi^{-1}(B_p)} \frac{|J_C|^2}{|f \circ \pi|^2} \ dVol_X$ converges if and only if the integrals on each such $U$ converge. Since $\pi$ is proper, the set $\pi^{-1}(p)$ is compact, and indeed, we
can check the convergence of the integral by computing the convergence on finitely many such open sets. Clearly, then, the log canonical threshold is finite, non-zero and rational: it will have the form $\frac{k_i+1}{b_i}$ coming from one of the charts, whichever of these is smallest over all charts.

The following exercise will test your understanding of the proof.

**Exercise 2.2.** Let $D = \{ f = 0 \}$ be a divisor on $\mathbb{C}^n$ and let $X \xrightarrow{\pi} \mathbb{C}^n$ be a log resolution. Write

$$\pi^* D = \sum_{i=1}^{t} b_i E_i, \quad K_{\pi} = \sum_{i=1}^{t} k_i E_i;$$

note that some of the $E_i$ above are the proper transforms of the components of $D$ (and $k_i = 0$ for those) and the remaining ones run through the exceptional divisors. Then the log canonical threshold of $D$ is

$$\min_{i=1}^{t} \{ \frac{k_i+1}{b_i} \}.$$

3. Singularities of log pairs

4. F-singularities

During the last two decades, a theory of “F-singularities of pairs” has flourished, inspired by the rich theory of pairs developed in birational geometry [?]. The idea to create tight closure theory for pairs was a major breakthrough, pioneered by Nobuo Hara and Kei-ichi Watanabe in [?]. Once defined, the theory of tight closure theory for pairs—including F-regularity, F-splitting and test ideals—rapidly developed in a long series of technical papers by the Japanese school of tight closure, including Hara, Watanabe, Takagi, Yoshida and others.

By *pair* in this context, we have in mind a normal irreducible scheme $X$ of finite type over a perfect field, together with either a $\mathbb{Q}$-divisor $\Delta$ (or an ideal sheaf $a$ raised to some fractional exponent)\[1\] In the geometric setting, an additional assumption—namely that $K_X + \Delta$ is $\mathbb{Q}$-Cartier—is usually imposed, because a standard technique involves pulling back to different birational models. One possible advantage to the algebra set-up is that it is not necessary to assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier for the definitions, although alternatives have also been proposed directly in the world of birational geometry as well; see [?]. See also [?].

**Definition 4.1.** Let $X$ be a normal F-finite variety, and $\Delta$ an effective $\mathbb{Q}$-divisor on $X$.

(a) The pair $(X, \Delta)$ is sharply Frobenius split (respectively locally sharply Frobenius split) if there exists an $e \in \mathbb{N}$ such that the natural map

$$\mathcal{O}_X \to F_p^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$$

\[1\]There are even triples $(X, \Delta, a^e)$ incorporating aspects of both variants.
splits as an map of sheaves of $\mathcal{O}_X$-modules (respectively, splits locally at each stalk).

(b) The pair $(X, \Delta)$ is globally (respectively, locally) F-regular if for all effective divisors $D$, there exists an $e \in \mathbb{N}$ such that the natural map

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

splits as an map of sheaves of $\mathcal{O}_X$-modules (respectively, splits locally at each stalk).

Remark 4.2. A slightly different definition of Frobenius splitting for a pair $(X, \Delta)$ was first given by Hara and Watanabe [?]. The variant here, which fits better into our context, was introduced by Karl Schwede [Sch10].

Theorem 4.3. [?] Let $(X, \Delta)$ be a pair where $X$ is a normal variety of prime characteristic and $\Delta$ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

(a) If $(X, \Delta)$ is a locally F-regular pair, then $(X, \Delta)$ is Kawamata log terminal.

(b) If $(X, \Delta)$ is a locally sharply Frobenius split pair, then $(X, \Delta)$ is log canonical.

4.1. Frobenius Splitting and Anti-Canonical Divisors.

Lemma 4.4. Let $X$ be a normal projective variety over a perfect field. Then we have

$$\mathcal{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \cong F^e_* \omega_X^{1-p^e}.$$

So we expect that globally F-regular schemes will admit many effective divisors in the linear systems $|(1 - p^e)K_X|$. Indeed, this property essentially characterizes globally F-regular varieties:

Theorem 4.5. [Smi00] [SS10] If $X$ is a globally F-regular projective variety of characteristic $p$, then $X$ is log Fano.

By log Fano we mean a normal projective variety which admits an effective $\mathbb{Q}$-divisor such that $-K_X - \Delta$ is ample, and the pair $(X, \Delta)$ has (at worst) Kawamata log terminal singularities. Alternatively, we can avoid the definition of KLT singularities as follows:

Theorem 4.6. [Smi00] [SS10] If $X$ is a globally F-regular projective variety of characteristic $p$, then

(a) There exists an effective $\mathbb{Q}$-divisor such that $-(K_X + \Delta)$ is ample.

(b) The pair $(X, \Delta)$ is F-regular.

Definition 4.7. Let $X$ be a normal variety of characteristic $p$, and let $\Delta$ be a $\mathbb{Q}$-divisor. We say that the pair $(X, \Delta)$ is F-regular if

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$^2$Kawamata log terminal singularity is usually defined in characteristic 0 using a resolution of singularities, but it can be defined in any characteristic as follows. A pair $(X, \Delta)$ consisting of a normal variety with an effective $\mathbb{Q}$-divisor is Kawamata log terminal if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and for all birational proper maps $\pi : Y \to X$ with $Y$ normal, choosing $K_Y$ so that $\pi_* K_Y = K_X$, each coefficient of $\pi^*(K_X + \Delta) - K_Y$ is strictly less than 1.
The proof of Theorem 4.6 constructs the $\mathbb{Q}$-divisor $\Delta$ as follows: Fix an ample divisor $D$. A splitting of Frobenius along $D$ is a map $F^e_\ast \mathcal{O}_X(D) \to \mathcal{O}_X$, which in turn can be viewed as a global section of $\text{Hom}_{\mathcal{O}_X}(F^e_\ast \mathcal{O}_X(D), \mathcal{O}_X) \cong \omega_X\cdot_p (-D)$. This gives rise to an effective divisor $D'$ in the linear system $((1 - p^e)K_X - D)$, and we set $\Delta = \frac{1}{p-1}D'$. Note that the construction very much depends on the characteristic, $p$.

Of course, our divisor clearly has the needed positivity but we must show that the singularities of $(X, \Delta)$ are controlled.

The converse of Theorem 4.6 fails because of irregularities in small characteristic. For example, the cubic hypersurface defined by $x^3 + y^3 + z^3 + w^3$ in $\mathbb{P}^3$ is Fano in every characteristic $p \neq 3$, but not globally F-regular nor even Frobenius split in characteristic two. However, it is globally F-regular for all characteristics $p \geq 5$.

**Theorem 4.8.** [Smi00, SS10] If $X$ is a log Fano variety of characteristic zero, then $X$ has globally F-regular type.

The converse to Theorem 4.8 is open. If $X$ has globally F-regular type, then in each characteristic $p$ model, the proof of Theorem 4.6 constructs a “witness” divisor $\Delta_p$ establishing that the pair $(X_p, \Delta_p)$ is log Fano. But $\Delta_p$ depends on $p$ and there is no a priori reason that there must be some divisor $\Delta$ on the characteristic zero variety which reduces mod $p$ to $\Delta_p$.

**Conjecture 4.9.** A projective globally F-regular type variety (of characteristic zero) is log Fano.

Conjecture 4.9 has been proved for surfaces [?] as well as for $\mathbb{Q}$-factorial Mori Dream spaces [?]. This raises the question: are globally F-regular type varieties (of characteristic zero) Mori Dream Spaces? Moreover, since log Fano spaces (of characteristic zero) are Mori Dream spaces by [?, Cor 1.3.2], the answer is necessarily yes if Conjecture 4.9 is true. What about in characteristic $p$?

**Question 4.10.** Assume that $X$ is globally F-regular. Is it true that the Picard group of $X$ is finitely generated? Is it true that the Cox ring of $X$ is always finitely generated?

Similar issues arise regarding the geometry of Frobenius split varieties:

**Theorem 4.11.** [SS10] If $X$ is a normal Frobenius split projective variety of characteristic $p$, then $X$ is log Calabi-Yau.

By log Calabi-Yau we mean that $X$ admits an effective $\mathbb{Q}$-divisor such that $(X, \Delta)$ is log canonical\(^3\) and $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to the trivial divisor.

Again, the converse fails because of irregularities in small characteristic. However, we do expect an analog of Theorem 4.8 to hold.

**Conjecture 4.12.** [SS10] If $X$ is a log Calabi-Yau variety of characteristic zero, then $X$ has Frobenius split type.

Conjecture 4.12 is known in dimension two [?] and for Mori Dream spaces [?].

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\(^3\)We define log canonical in characteristic $p$ similarly to how we defined klt singularities.
Example 4.13. Grassmannians of any dimension and characteristic are globally F-regular. Indeed, the homogeneous coordinate ring for the Plücker embedding of any Grassmannian is F-regular [LRPT06]. More generally, all Schubert varieties are globally F-regular [LRPT06]. A normal projective toric variety (of any characteristic) is globally F-regular, since a section ring given by a torus invariant ample divisor will be generated by monomials, hence F-regular [LRPT06].

Similarly, there are global versions: Theorem 4.8 and 4.11 also hold for “pairs.” See SS10. In characteristic zero, the converse of (1) holds, as does its global analog. The local and global converses of (2) are conjectured; this appears to be a difficult problem with deep connections to arithmetic.

We can think of F-regularity as a “characteristic $p$ analog” of Kawamata log terminal singularities, and (at least conjecturally) Frobenius splitting as a “characteristic $p$ analog” of log canonical singularities. The analogy runs deep: F-pure thresholds become “characteristic $p$ analogs” of log canonical thresholds [LRPT06], test ideals become “characteristic $p$ analogs” of multiplier ideals [LRPT06], centers of sharp F-purity become “characteristic $p$ analogs” of log canonicity [LRPT06], F-injectivity becomes a “characteristic $p$ analog” of Dubois singularities [Sch09].

4.1.1. Possible Applications to the Minimal Model Program in characteristic $p$. Recently, attention has turned to solving the minimal model program in prime characteristic, where a big obstruction is the failure of vanishing theorems. There is hope that the Frobenius splitting and tight-closure inspired definitions will help overcome this difficulty. For example, it is not even known in characteristic $p$ whether klt singularities are Cohen-Macaulay, even when resolution of singularities is assumed [LRPT06]. Perhaps F-regularity is the “right” class of singularities to consider in prime characteristic instead? As another example, the test ideal is better than the multiplier ideal at capturing some of the subtleties in prime characteristic, for example, under pull back under wildly ramified mappings [LRPT06]. The world of F-singularities is beginning to get implemented in the minimal model program (see e.g. [LRPT06]), but the final outcome of this endeavor is not yet clear. Another place where Frobenius techniques have been helpful is in effective generation of adjoint bundles; this goes back to [Smi97a] which is reproved in dual format in [Smi97a], and generalized recently in [LRPT06]. See also [LRPT06].
Bibliography


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