Algebra 2: Harjoitukset 1.

A. Definition: A point \( p \) in the Cartesian plane is **rational** if \( p = (x_1, y_1) \) with \( x_1, y_1 \in \mathbb{Q} \). A line in the Cartesian plane is **rational** if it is determined by (passes through) two rational points.

1. Prove that a line in the Cartesian plane is rational if and only if it can be given by an equation \( ax + by = c \) where \( a, b, c \in \mathbb{Q} \).
2. Prove that the intersection of two rational lines in the Cartesian plane is a rational point (or empty).

B. Fix any real extension field \( K \) of \( \mathbb{Q} \). We say that a point \( p \) in the Cartesian plane is **\( K \)-rational** if its coordinates \( (x_1, y_1) \) satisfy \( x_1, y_1 \in K \). A **\( K \)-rational line** is a line determined by two \( K \)-rational points. Prove that the intersection of any two \( K \)-rational lines is a \( K \)-rational point (or empty).

C. Definition: A circle in the Cartesian plane is **rational** if its center is a rational point and it passes through a rational point. Does the intersection of a rational circle with a rational line in the Cartesian plane always consist of rational points? Explain. What do you think a \( K \)-rational circle should be (where \( K \) is a real extension field of \( \mathbb{Q} \))? What can be said about the intersection of a \( K \)-rational circle with a \( K \)-rational line?

D. Let \( F \) be the set of words \( \{ \text{even}, \text{odd} \} \). Define binary operations + and \cdot on \( F \) in the obvious way: \( \text{even} + \text{odd} = \text{odd}, \text{even} \cdot \text{odd} = \text{even} \) etc. Construct an addition table and a multiplication table for these operations. Does this set form a field with these operations? What are the additive and multiplicative identities?

E. Using only the axioms of the field definition, prove that the additive and multiplicative identities of a field are unique. Prove also that the additive inverse and multiplicative inverse of each (non-zero) element is unique.

F. Compute the degrees of the following extensions. Exhibit an explicit vector space basis in each case.

1. \( \mathbb{R} \subset \mathbb{C} \).
2. \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{17}) \).
3. \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{17}, \sqrt{19}) \).
4. \( \mathbb{Q} \subset \mathbb{Q}(\sqrt{17}, \sqrt{19}, \theta) \), where \( \theta \) is a complex number such that \( \theta^2 \in \mathbb{Q}(\sqrt{17}, \sqrt{19}) \).
5. \( \mathbb{Q} \subset K \) where \( K \) is the extension of \( \mathbb{Q} \) obtained by adjoining all third roots of unity, that is, \( K = \mathbb{Q}(\{e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}, 1\}) \). [Hint: For a complex number \( z \) on the unit circle in \( \mathbb{C} \), \( \bar{z} = z^{-1} \).]

BONUS: Let \( K = \mathbb{Q}(S) \) where \( S \) is the set of all the complex fifth root of unity. Prove that \( [K : \mathbb{Q}] = 4 \). You may assume that the polynomial \( x^4 + x^3 + x^2 + x + 1 \) is irreducible. [Hint: \( e^{\frac{2\pi i}{5}} \) satisfies \( x^5 - 1 \), which can be factored.]

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1 Meaning that \( \mathbb{Q} \subset K \subset \mathbb{R} \).