Algebra 2: Harjoitukset 10. Due November 23

A. Let $K \subset L$ be a normal field extension with Galois group $G$. Take any $\beta \in L$. Show that

$$\sum_{\sigma \in G} \sigma(\beta) \quad \text{and} \quad \prod_{\sigma \in G} \sigma(\beta)$$

are both in $K$.

B. Let $K \subset L$ be a normal field extension of degree $p$, prime (both subfields of $\mathbb{C}$).

(1) Prove that $L = K(\beta)$ for any $\beta \in L \setminus K$.

(2) Prove that (with $\beta$ as in (1)), the extension $K_p \subset K_p(\beta)$ is also of degree $p$, where $K_p$ is the splitting field of $x^p - 1$ over $K$.

(3) Prove that the Galois group of $K_p(\beta)$ over $K_p$ is cyclic of order $p$.

(4) Let $\phi$ be a generator for the Galois group in (3). Define $\alpha = \sum_{n=0}^{p-1} \eta^{-n} \phi^n(\beta)$, where $\eta = e^{2\pi i/p}$. Show that $\phi(\alpha) = \eta \alpha$.

(5) Prove that $\alpha^p \in K_p$.

(6) Find the minimal polynomial of $\alpha$ over $K_p$.

(7) Show that every element of $K_p(\beta)$ is expressible in radicals over $K$.

(8) Conclude that if the Galois group of a polynomial $f \in K[x]$ is both solvable and simple, then its roots are expressible in radicals over $K$. This completes the missing part of the proof of the theorem from Lecture of November 14: A polynomial $f \in K[x]$ has roots expressible in radicals over $K$ if and only if its Galois group is solvable.

C. Prove that a (not necessarily normal) subgroup of a solvable group is solvable. [Hint: Noether’s isomorphism theorems.] Use this to show that every polynomial over $K$ of degree 4 or less has roots expressible in radicals over $K$ (here $K$ is any field containing $\mathbb{Q}$).

D. Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial of degree five with exactly one real root. Prove that its Galois group of order at least 10.

E. For each polynomial below, decide whether or not its roots are expressible in radicals over $\mathbb{Q}$.

(1) $x^6 - 8$

(2) $(x^4 + x^3 - 6x^2 + 2x^2 - 26x + 7)(x^3 - 4x^2 + 5x + 9)$

(3) $x^6 - 5x^3 + 6$

(4) $x^5 - 5x^4 + 17$. 